

# Fundamental Ambiguity in the Definition of Vertical Motion

Geoffrey Blewitt

Mackay School of Earth Sciences and Engineering, University of Nevada, Reno, USA.

**Abstract.** Vertical motion is generally described as motion normal to some defined horizontal surface or family of surfaces. Such a surface can be constructed geometrically or gravitationally. In the case of a geometrical surface, the ellipsoid is the shape of choice. In the case of a gravitational surface, it needs to be specified which masses are to be included in the definition of the center-of-mass origin, and whether the surface evolves in time along with the configuration of mass (as in a dynamic equipotential) or whether it is static (as in a reference geoid). Vertical motion is therefore not an absolute concept, but rather one of convenience and convention. Even if properly defined, the ability to realize vertical displacement coordinates may be limited by uncertainty in the relationship between observed quantities and the frame definition. Thus discrepancies between two analysis groups' results on vertical station velocities may not be a result of systematic error, but rather may simply reflect inability to realize the same reference frame.

**Keywords.** Vertical, reference frame, ellipsoid, gravity, gravitation, geoid, equipotential, sea level.

## 1 Issues for Discussion

Vertical motion is an ambiguous term, in the sense that vertical displacement coordinates depend on the adopted conventions of the reference frame and coordinate system, and the ability to realize such a reference frame in practice. This paper addresses the roots of this ambiguity by posing the question, "what is vertical motion?" We identify different classes of defined vertical motion, which are based on physical, geometric, and hybrid models.

By analyzing this ambiguity, we investigate to what extent differences between these models are predictable. Part of the answer of course lies with reference frames and reference system conventions.

Finally, we address the implications for ambiguity in vertical motion, including its effect in inter-comparison of vertical motion results, and on scientific interpretation of vertical motions.

## 2 What is Vertical Motion?

Vertical motion is generally defined as motion that is normal to some defined horizontal surface. Thus it is entirely defined by the horizontal surface itself.

### 2.1. Physical Model

In a physical model, the horizontal surface is generally taken as an equipotential surface. More correctly, an infinite family of equipotentials define the direction of vertical motions everywhere. The direction of vertical motion follows a "plumb-line" trajectory, which is generally curved.

Despite its physical nature, there are many ways to define vertical motion within this class of model. Most fundamentally, "equipotential" is not an absolute concept, as it depends on the local relativistic frame. The equivalence principle states that acceleration cannot be distinguished from gravity, hence equipotential surfaces be different in frames that accelerate relative to one another.

One obvious cause of acceleration is rotation. Co-rotating frames therefore "see" different equipotential surfaces. Since Earth rotation is far from constant, this presents a problem. A conventional solution is to adopt some defined constant angular velocity for Earth rotation, thus enforcing a conventional centrifugal contribution to the equipotential. In this case, it is typical to define "gravity" as being equal to "gravitation" plus the conventional (rather than actual) "centrifugal" term. Gravitation is here understood to be in a quasi-inertial local-Earth frame, co-orbiting the Sun along with the Earth system's center of mass, and so is the Earth system's gravity field as sensed by orbiting satellites, after accounting for solar system gravitational forces. However this is only one possibility.

Note that other choices of origin and rotation create additional fictitious (non-inertial) forces and so have different equipotential surfaces. Other possibilities are frames that co-rotate with no-net rotation with respect to the Earth's surface, which raises the further issue of whether this condition is imposed as a long term average, or instantaneously as Earth rotation varies. The dynamic interpretation of

vertical motion therefore strictly requires accounting for the choice of frame in this respect. In most cases the choice may not matter for practical purposes, however the theoretical distinctions must be kept in mind if we are to catch those few instances of scientific interpretation where it does actually matter.

## 2.2. Geometric Model

In a geometric model, the horizontal surface is generally taken as a conventional ellipsoid (sometimes referred to as a “datum”). In this system, vertical motion is defined as motion along a straight line that intersects the ellipsoid at right angles. This is very different than in the physical model, where pure vertical motion does not generally follow a straight line.

There are as many ways to define vertical motion within this class of model as there are possible sets of ellipsoids. The typical convention is to adopt a perfectly circular equator, in which case the ellipsoid’s internal geometry is defined by two parameters: the radius of the equator, and the flattening factor. The external geometry (defining the location and orientation of the ellipsoid in space) is defined by an additional six parameters: three for the origin, two to define the orientation of the polar axis of symmetry, and one to define the origin of longitude (the prime meridian). Note that the ellipsoid’s external geometry, including its center, is in practice defined implicitly by a defined set of station coordinates and their evolution in time. This is called the “conventional terrestrial reference frame”, or CTRF.

Even if we can all agree on a convention for the ellipsoid’s internal geometry, it is more difficult to be consistent in the definition of the external geometry, because this requires consistency in the various CTRFs. For example, different techniques use different stations, and so convention would require the adoption of accurate “local ties” between (nearby) co-located stations. There would also need to be consistency in the time evolution of the station coordinates.

In any situation where two groups have adopted a different set of stations defining a CTRF, the definition of vertical motion will of course be different. Differences in vertical motion between two different frames are typically significant (much larger than inherent measurement precision), especially if there is relative motion between the origins of the two frames. This is major source of confusion when comparing vertical velocities of individual sites between two different techniques (and even within the

same technique by two different analysis groups).

## 2.3. Hybrid Model

In a hybrid model, the horizontal surface is generally taken as a conventional ellipsoid, however the center of the ellipsoid is typically taken as the Earth center of mass (as a function of time). Defining the center in a physical way helps to resolve at least one source of ambiguity, although in practice different techniques produce slightly different solutions for the location of the Earth’s center of mass. Another problem here is that the origin is typically taken as the center of mass of the whole Earth system (to which satellite orbits are sensitive), however the solid Earth itself (approximately an ellipsoid) must move with respect to this point as mass is redistributed in its fluid envelope. So we have a situation where in reality, the ellipsoid should really be moving, but in a typical CTRF this is not modeled (or not adequately modeled). The definition of vertical motion in the hybrid model will depend on whether or not this “geocenter motion” is modeled, or whether or not models specify time evolution of the solid Earth’s deforming surface.

## 3 Analyzing the Ambiguity

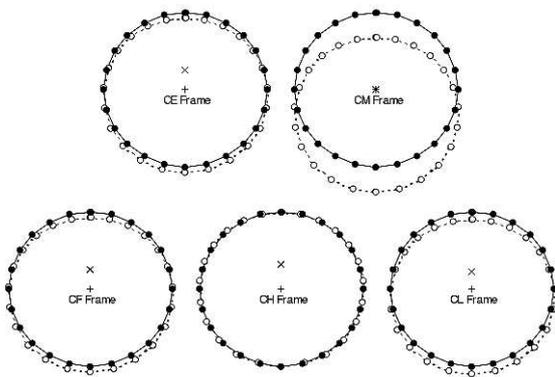
From the above discussion, it is clear that “vertical motion” is not an absolute concept. From this conclusion we can logically deduce the following corollaries. Since vertical and horizontal motion as defined above span 3-dimensional space, therefore horizontal motion too is not absolute. From this we can infer that vertical motion in one reference system can appear horizontal in another, and vice versa.

### 3.1. Reference Frame Issues

One might at this point naturally ask whether or not these apparent differences are predictable, and in what cases are the differences accentuated or extreme. Logically, the most extreme case is where pure vertical motion in one system appears as pure horizontal motion in another. As ridiculous as this may appear, there is actually such a case that corresponds to a very real physical phenomenon!

An explanation of this phenomenon requires a fundamental consideration of reference frame theory, as explained in *Blewitt* [2003], with further discussion in *Blewitt and Clarke* [2003], where it was shown how a perfect sphere deforms to another perfect sphere when subjected to a degree-1 spheri-

cal harmonic surface load. Moreover, the sphere is displaced in inertial space due to conservation of momentum, such that the center of mass of the solid Earth moves in response to a change in center of mass of the surface load. Depending on the choice of reference frame, the resulting vector field of surface displacements can appear either purely vertical, or purely horizontal. This is illustrated in Figure 1, which appears in *Blewitt* [2003]:



**Fig. 1** Degree-1 deformation of the solid Earth. The spherical shape of the Earth remains unchanged. However points on the Earth's surface do move relative to each other. Depending on the choice of frame, the deformed sphere may appear to have translated with respect to its initial configuration. The symbol X denotes the center of mass of the entire system, CM. Also shown are the center of mass of the solid Earth, CE; the 3-D center of figure of the Earth's surface, CF; the center of height figure of the Earth's surface, CH; and the center of lateral figure of the Earth's surface, CL.

Note that in Figure 1, the deformation is identical for all five frames shown. However, the vectors describing the displacement of stations on the Earth's surface are completely different. For example, the displacement field in the CH frame is entirely horizontal, whereas in the CL frame it is entirely vertical!

In practical terms, the reference frame most commonly used by solid Earth modelers is CE, center of mass of the solid Earth. However this is never a frame of reference that can be realized by space geodesy. In GPS practice, for example, the CF frame is most commonly used. The CM frame is possible, but less precisely realized in GPS due to errors in orbit models. In SLR, the CM frame is a natural frame because the laser ranging satellites have well behaved (modeled) orbits.

A specific model of the solid Earth might not explicitly explain the reference frame. For example, the model might simply assume a priori that the deformation is purely vertical (e.g., atmospheric

loading models that use a proportional relationship between vertical displacement and barometric pressure). In that case, the CL frame is implicitly assumed. Another model might assume a priori that surface motions are entirely horizontal (e.g., rigid plate rotations). In this case, the CH frame is implicitly assumed.

Given that there are so many possible frames that are being used in either modeling or in space geodetic analysis, it is imperative that care be taken to account for those frame differences when attempting to interpret station motions. The "observed" station displacements may appear to disagree with a model, but actually they may be consistent once the reference frame differences are taken into account.

One way to account for reference frame differences is to apply exactly the same reference frame procedure to the modeled displacements as to the observed displacements. There are also other methods. For example, when applying the Green's function approach to modeling atmospheric loading, *Blewitt* [2003] shows that the degree-1 load Love numbers are frame dependent. Table 1 lists values that can be used for the various suite of frames described above, derived from load Love numbers computed originally in the CE frame by *Farrell* [1972].

**Table 1.** Degree-1 load Love numbers

Frame Type	Height $h_1$	Lateral $l_1$	Potential $1+k_1$
CE	-0.290	0.113	1
CM	-1.290	-0.887	0
CF	-0.269	0.134	1.021
CL	-0.403	0	0.887
CH	0	0.403	1.290

A different type of reference frame issue relates to the scale of the frame. If the mass of the Earth is conserved, there is no degree-0 component of deformation. That is, the average radius of the Earth does not change in time. Meteoritic mass accumulation is far too small to have any detectable effect on degree-0 deformation. However, some reference frames do not constrain degree-0 deformation to be zero, resulting in time variation in the scale of the global network.

When realizing a reference frame, typically a transformation is applied using the usual seven Helmert parameters plus their seven time derivatives. Alternatively, the scale and scale rate parameters might not be included in this transformation. Of course, vertical motion would appear to be

different between these two methods of frame realization.

### 3.2. Sea Level and Geoid Ambiguity

The term “sea level change” can be ambiguous. It should always be made clear as to whether we are referring to relative sea level or geocentric sea level. Relative sea level is the height of the sea surface above the ocean bottom (which itself deforms due to the changing ocean load). Relative sea level is therefore a physical concept. Geocentric sea level is change in the height of the sea surface with respect to a CTRF, which therefore is very sensitive to the type of frame and the definition of the frame’s origin. Geocentric sea level therefore is a more abstract concept and requires careful definition in its scientific application.

The geoid is often specified as the reference surface used to define vertical motion. However different authors use the word “geoid” in different ways. Classically, the geoid is the equipotential surface that most closely represents the mean sea surface, and is fixed in time. Some authors may use “geoid” to indicate this equipotential surface as it varies in time. This might be better termed the “deformed geoid” or “time varying geoid.” And some authors might use “geoid” to indicate a time-varying surface that is allowed to vary its gravitational potential to account for time-variation in the volume of water in the ocean. This might be better termed the “sea surface equipotential”. Indeed, the geopotential height of the sea surface changes with ocean volume, whereas the geopotential height of the deformed geoid should be zero by definition, independent of what is happening with the ocean. Clearly, the definition of vertical motion is different depending on which surface is chosen. Care should be taken when interpreting data in the context of models that use such concepts of sea level and geoid.

## 4 Conclusions

When performing scientific interpretation in geodesy, the models and observational data might fall in two categories: (1) where vertical is defined to be normal to the gravity equipotential, and (2) in the case of GPS, where vertical is defined to be normal to a geocentric ellipsoid. A third category might include models where not much attention has been given to define vertical very clearly, and so might limit scientific interpretation. For example, some “sea level rise” calculations due to net influx of wa-

ter into the oceans might implicitly consider the rise to be “vertical”. Sometimes sea level change is treated as an addition of a uniform layer. This ignores self-gravitation of the redistributed mass. It also ignores self-gravitation of the solid Earth deformed by the redistributed surface mass.

For case (1) one should be careful about the definition of the origin of the frame. Choices might include the center of mass of the solid Earth (only), the center of mass of the entire Earth system, or the center of figure of the Earth’s surface. Then there is the question of which equipotential surface is being used to define vertical motion: is it the “classical” geoid for example, or is it the time varying sea surface that is being called the “geoid”?

In conclusion, we have discussed how vertical motion is a relative concept. Gravitationally horizontal processes can induce apparent vertical motion. Also, gravitationally vertical processes might not be apparent in geodetic vertical data. The implications of this are that (1) inter-comparisons are inherently difficult, especially if the reference frame definition is ambiguous; (2) improved reference frames are needed, for example, that take into account seasonal mass redistribution, its effect on degree-1 deformation, and therefore its effect on the frame origin; (3) scientific interpretation should be mindful of vertical ambiguity, and should seek to carefully clarify what is meant by “vertical”.

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