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Geodetic network optimization for geophysical parameters

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Abstract. The first order design problem in geodesy is generalized here, to seek the network configuration that optimizes the precision of geophysical parameters. An optimal network design that satisfies intuitively appropriate criteria corresponds to minimizing the sum of logarithmic variances of eigenparameters. This is equivalent to maximizing the determinant of the design matrix, allowing for closed-form analysis. An equivalent expression is also given specifically for square root information filtering, to facilitate numerical solution. Appropriate seeding of numerical solutions can be provided by exact analytical solutions to idealized models. For example, for an ideal transform fault, simultaneous resolution of both the locking depth *D* and location of the fault is optimized by placing stations at $\pm D/\sqrt{3}$ (~9 km) from the a priori fault plane. In a two-fault system, the resolution of slip partitioning is optimized by including a station midway between faults; however resolution is fundamentally limited for fault separation <2*D* (~30 km).

1. Introduction

"First order design" in geodesy is the problem of finding the schedule of observations that optimizes the precision of estimated network geometry [*Vanicek and Krakiwsky*, 1982]. To today's investigator using the Global Positioning System (GPS), optimizing coordinate precision through improved network design is no longer a serious consideration [*Zumberge et al.*, 1997]. In contrast, this paper addresses network design that optimizes geophysical parameter precision. This is a timely topic, given new initiatives such as the Plate Boundary Observatory, and the general rapid growth of GPS networks for geophysical research [*Segall and Davis*, 1997].

For a given network, the Network Inversion Filter by *Segall and Matthews* [1997] is state-of-the-art, addressing geophysical parameter resolution and accounting for correlated errors. This might be incorporated into a heuristic approach to network design, but as of yet, there is no systematic method to design networks for geophysical investigations. A complementary and potentially insightful approach is to formalize the design problem, enabling closed-form analysis. This paper derives results for examples of ideal models, which can then be used to seed algorithmic searches for solutions to more sophisticated geophysical models and filtering techniques. An optimization functional specific to square root information filtering (SRIF) is presented here, which would facilitate efficient stochastic modeling.

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2. Formalizing the Problem

2.1. Geodetic Model

Given *n* stations at positions $\mathbf{r}_i$ for i=1,2,3…m, let geodesy provide estimates of *m* station velocities $\mathbf{u}_i$ with the $m \times m$ co variance matrix **C**. The relevant velocity vectors $\mathbf{u}_i$ may have 1, 2, or 3 components. For least-squares estimation of the geophysical parameters, $\mathbf{u}_i$ will be used as input data, and the inverse of **C** will be the fully populated data weight matrix, **W**. Although the data type is station velocity, the following could be adapted to time series of displacements, thus accommodating models with explicit time dependence.

2.2. Geophysical functional model

Assume that each station velocity vector $\mathbf{u}_i$ can be written as an analytical function of station position $\mathbf{r}_i$ and geophysical parameters g_k for k=1,2,3…n:

$$\begin{aligned} \mathbf{u}_1 &= \mathbf{G}(\mathbf{r}_1; g_1, g_2, \cdots g_n) \\ \mathbf{u}_2 &= \mathbf{G}(\mathbf{r}_2; g_1, g_2, \cdots g_n) \\ &\vdots \\ \mathbf{u}_m &= \mathbf{G}(\mathbf{r}_m; g_1, g_2, \cdots g_n) \end{aligned} \tag{1}$$

The explicit dependence on only the local station's position assumes that the zero order design (i.e., the velocity datum) is in accordance with the geophysical model. Specification of an appropriate velocity datum is usually achievable using global geodetic networks. While this is not essential to the physics, it does allow for the simple form of equation (1).

2.3. Geophysical Parameter Precision

The following shows formally how parameter precision depends on network design. In preparation for weighted least squares analysis, equation (1) is linearized about provisional parameter values (denoted by tildes):

$$\mathbf{u}_i = \mathbf{G}(\mathbf{r}_i; \tilde{g}_1, \tilde{g}_2, \cdots \tilde{g}_n) + \mathbf{A} \cdot \left(\mathbf{g} - \tilde{\mathbf{g}}\right) \tag{2}$$

where the design matrix **A** ($m \times n$) contains partial derivatives of the velocity model with respect to the geophysical parameters, evaluated at the provisional values

$$\mathbf{A}_{ik} = \left. \frac{\partial \mathbf{G}(\mathbf{r}_i; g_1, g_2, \cdots g_n)}{\partial g_k} \right|_{\tilde{g}_1, \tilde{g}_2, \cdots \tilde{g}_n} \tag{3}$$

The variances and covariances of the estimated parameters are given by the matrix

$$\mathbf{P}\left(\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_m; \tilde{g}_1, \tilde{g}_2, \cdots, \tilde{g}_n\right) = \left(\mathbf{A}^{\mathrm{T}} \mathbf{W} \mathbf{A}\right)^{-1} \tag{4}$$

This emphasizes **P** as a function of all station positions and provisional parameters.

2.4. Generalized First Order Design

Matrix **P** characterizes the precision of the parameters. Let us characterize overall precision by a single number, which can be minimized by varying the station locations. Consider first an optimization functional J, which is some (as yet undefined) function of the covariance matrix:

$$J\{\mathbf{P}(\mathbf{r}_1,\mathbf{r}_2,\cdots,\mathbf{r}_m;\tilde{g}_1,\tilde{g}_2,\cdots,\tilde{g}_n)\} \tag{5}$$

For each trial network defined by the set $\{\mathbf{r}_i \mid i=1,2\ldots,m\}$, expression (5) would yield a single number for assessment.

Now let us vary this functional by perturbing every station position by a small amount $\boldsymbol{\delta}\mathbf{r}_i$:

$$\delta J\{\mathbf{P}(\mathbf{r}_1,\mathbf{r}_2,\cdots,\mathbf{r}_m;\tilde{g}_1,\tilde{g}_2,\cdots,\tilde{g}_m)\} = \frac{\partial J}{\partial \mathbf{r}_1}\boldsymbol{\delta}\mathbf{r}_1 + \frac{\partial J}{\partial \mathbf{r}_2}\boldsymbol{\delta}\mathbf{r}_2 \cdots + \frac{\partial J}{\partial \mathbf{r}_m}\boldsymbol{\delta}\mathbf{r}_m \tag{6}$$

The set of station positions $\breve{\mathbf{r}}_i$ that minimizes this functional is such that variation is zero for any arbitrary set of station perturbations. For every station i, the solution must satisfy

$$\mathbf{K}_i(\breve{\mathbf{r}}_1,\breve{\mathbf{r}}_2,\cdots,\breve{\mathbf{r}}_m;\tilde{g}_1,\tilde{g}_2,\cdots,\tilde{g}_n) = \frac{\partial J}{\partial \mathbf{r}_i} = 0 \tag{7}$$

This is a system of m equations, each with a different vector function $\mathbf{K}_i$ of each station, for which we seek m unknown positions $\breve{\mathbf{r}}_1,\breve{\mathbf{r}}_2,\cdots,\breve{\mathbf{r}}_m$. The problem of generalized first order design, therefore, is formally stated as finding the set $\{\mathbf{r}_i \mid i=1,2\ldots,m\}$ that satisfies (7), for some specific functional J. To proceed, we must make a specific choice of functional.

3. Optimization Functional

3.1. General Considerations

It is by no means obvious which optimization functional to choose. It is useful to begin by stating general principles to guide selection.

3.1.1. Geophysical Interest. Let us assume all geophysical parameters are of interest, and should be included in the functional. The solution to the m equations in (7) generally (but not necessarily) depends on the provisional parameters; yet these parameters will be estimated. This means that we might need reasonable a priori knowledge, e.g., a hypothesis.

3.1.2. Independence of Units. Geophysical parameters generally have different units, but the solution should not depend on the choice of units. (The trace of the covariance does not satisfy this criterion, and only works for classical first order design because all coordinates happen to have the same units.)

3.1.3. Independence of Parameter Basis. The linearized geophysical model can be reparameterized by linear combinations of the original parameters, giving identical solutions. For example, slip rates on two faults might be reparameterized in terms of the difference in slip, and sum in slip. Network design should be independent of such a change in parameter basis.

3.1.4. Analytical Simplicity. Starting with a functional that has intuitively appropriate properties, we seek to derive an equivalent, simpler functional that always leads to the same solution. Even so, analytical solution might not be possible for more complex cases. One way forward is to seek the solution numerically, starting with an approximate answer. For this it would be useful to initialize the search using solutions to simpler problems. Therefore, it is useful to choose a functional that is amenable to analytical solution.

3.2. Specific Choice of Functional

The specific choice of functional proposed is the sum of logarithmic variances associated with the geophysical eigenparameters. The eigenparameters result from an orthogonal transformation of the original parameters, where the transformation matrix is formed from eigenvectors of **P**. The eigenparameters have a diagonal covariance matrix **P'**, whose elements are eigenvalues λ_k of **P**. These eigenvalues are variances associated with the eigenparameters. They generally depend on all station positions and provisional parameters

The rationale for using eigenparameters is that they are uncorrelated (**P'** is diagonal), hence their variances summarize all information on precision. Defining the functional in terms of eigenvalues ensures independence of parameter basis (3.1.3). The logarithms ensure unit independence (3.1.2). The sum is taken to include all parameters (3.1.1):

$$J = \sum_{k=1}^{n} \ln \lambda_k \tag{8}$$

3.3. Functional Equivalence

It is now shown that this functional is equivalent to the determinant of the design matrix **A** of equation (3). Firstly,

$$J = \sum_{k=1}^{n} \ln \lambda_k = \ln\left(\prod_{k=1}^{n} \lambda_k\right) \Leftrightarrow \prod_{k=1}^{n} \lambda_k = \det \mathbf{P'} \tag{9}$$

where the symbol $\prod$ denotes multiplication. The symbol $\Leftrightarrow$ denotes functional equivalence, in the sense that the values of parameters that minimize equivalent functionals are identical.

Now we use the property that the determinant of $\mathbf{P'}$ is unchanged under an orthogonal transformation $\boldsymbol{\Gamma}$, such as the one that transformed it from the original covariance matrix **P**

$$\begin{aligned} J &= \det \mathbf{P'} = \det(\boldsymbol{\Gamma}\boldsymbol{\Gamma}^{\mathbf{T}})\det \mathbf{P'} = \det(\boldsymbol{\Gamma}\mathbf{P'}\boldsymbol{\Gamma}^{\mathbf{T}}) \\ &= \det \mathbf{P} \end{aligned} \tag{10}$$

At this point, note that if we change the units of any parameter, it is equivalent to multiplying $\det \mathbf{P}$ by a constant. Hence the functional is independent of parameter units (3.1.2).

The algebra of functional equivalence simplifies (10) even further:

$$J = \det \mathbf{P} \Leftrightarrow \sum_{k=1}^{n} \ln \lambda_k = -\sum_{k=1}^{n} \ln \lambda_k^{-1} \Leftrightarrow -\det \mathbf{P}^{-1} \tag{11}$$

That is, the minimum of $\det \mathbf{P}$ corresponds to the maximum of $\det \mathbf{P}^{-1}$. Now consider the minimal case where the number of velocity data m equals the number of parameters n:

$$J = -\det \mathbf{P}^{-1} = -\det(\mathbf{A}^{\mathbf{T}}\mathbf{W}\mathbf{A}) \Leftrightarrow -|\det \mathbf{A}| \tag{12}$$

It is assumed here that the data weights are independent of station location. This is a reasonable a priori assumption for regional GPS networks. From equation (12) and (7) we have

$$\frac{\partial}{\partial \mathbf{r}_i} \det \mathbf{A} = 0 \tag{13}$$

This is the key equation of the paper. Equation (13) is the specific form of (7). Solutions to (13) should be checked to ensure $|\det \mathbf{A}|$ is at its maximum. Unlike covariance analysis, no matrix inversion is involved, which facilitates closed-form analysis (3.1.4).

3.4. Numerical Solution by Square Root Information Filtering

Numerical solution using the square root information filter (SRIF) would allow for spatially correlated geodetic data, and process noise for stochastic temporal variation in the parameters [*Bierman*, 1997]. SRIF is routinely used for GPS data reduction [*Zumberge et al.*, 1997], and could be implemented for sophisticated fault modeling. For SRIF, (12) reduces to

$$J = -\left|\prod_{k=1}^{n} R_{kk}\right| = -\left|\sum_{k=1}^{n} \ln R_{kk}\right| \tag{14}$$

where R_{kk} are diagonal elements of the (triangular) square root information matrix.

Processing time would be dominated by iteration of the filter to minimize equation (14) numerically. Note that, to find the optimal solution, there is no need to compute the solution (i.e., no need to invert the SRIF matrix). The sum of logarithms is numerically essential. To aid convergence, it is useful to seed the algorithm with approximate solutions, which may be derived by exact solution to simpler, ideal cases. When seeding is not obvious, a hybrid simplex- Monte-Carlo algorithm has proved useful [*Clarke et al.*, 1997] for fault parameter inversion, and should be applicable here.

4. Example: The Ideal Transform Fault

4.1. Unknown Slip Rate

Consider the elastic dislocation model for an infinite transform fault locked down to depth D with slip rate at depth u_0. Taking the x axis to lie normal to the fault plane, the velocity of a station i at coordinate x_i relative to the far left field of a left-lateral strike-slip fault is

$$u_i = \mathbf{G}(x_i; u_0, x_0, D) = \frac{u_0}{\pi}\arctan\frac{x_i - x_0}{D} + \frac{u_0}{2} \tag{15}$$

where the fault is located at x_0. Provisionally we set $x_0 = 0$.

Let us find the position of a station (i=1) that optimizes the slip rate u_0, assuming velocity is expressed relative to some other station in the far left field. In this case, (13) becomes

$$0 = \frac{\partial}{\partial x}\frac{\partial \mathbf{G}(x;D)}{\partial u_0} = \frac{1}{\pi D[(x/D)^2 + 1]} \tag{16}$$

The solution to (16) that maximizes $|\det \mathbf{A}|$ is:

$$(x/D) \to +\infty \tag{17}$$

The optimal network therefore has two stations far on opposite sides of the fault.

4.2. Unknown Depth of Locking

Let us find the position of a station (i=1) that optimizes D, assuming both slip rate u_0 (using stations in the far field either side of the fault) and the fault location x_0. In this case,

$$0 = \frac{\partial}{\partial x}\frac{\partial \mathbf{G}(x;D)}{\partial D} = \frac{u_0}{\pi\left(x^2 + D^2\right)^2}(x - D)(x + D) \tag{18}$$

Solving (18),

$$x = \pm D \tag{19}$$

The optimal network therefore has three stations, one at distance D (~15 km) either side of the fault trace, the other two at many locking depths on opposite sides of the fault.

4.3. Unknown Depth of Locking and Fault Location

Let us solve for the positions of two stations (i=1,2) that optimize parameters x_0 and D, assuming u_0 is resolved:

$$\begin{aligned} 0 &= \frac{\partial}{\partial x_i}\left(\frac{\partial G(x_1)}{\partial x_0}\frac{\partial G(x_2)}{\partial D} - \frac{\partial G(x_2)}{\partial x_0}\frac{\partial G(x_1)}{\partial D}\right) \\ &= \frac{\partial}{\partial x_i}\left(\frac{u_0{}^2 D(x_2 - x_1)}{\pi^2\left(x_1^2 + D^2\right)\left(x_2^2 + D^2\right)}\right) \end{aligned} \tag{20}$$

Discarding constant terms gives the pair of equations

$$\begin{aligned} 0 &= \frac{\partial}{\partial x_1}\left(\frac{x_2 - x_1}{x_1^2 + D^2}\right) = \frac{x_1^2 - 2x_1x_2 - D^2}{\left(x_1^2 + D^2\right)^2} \\ 0 &= \frac{\partial}{\partial x_2}\left(\frac{x_2 - x_1}{x_2^2 + D^2}\right) = \frac{-x_2^2 + 2x_1x_2 + D^2}{\left(x_2^2 + D^2\right)^2} \end{aligned} \tag{21}$$

Solving these simultaneous equations,

$$x_1 = \pm\frac{D}{\sqrt{3}} \quad \text{and} \quad x_2 = \mp\frac{D}{\sqrt{3}} \tag{22}$$

The optimal network therefore has four stations distributed symmetrically around the suspected fault trace: two at many locking depths from the fault, and two at 0.6D (~ 9 km).

4.4. Two-Fault Systems

Determining how slip at depth might be distributed between parallel transform faults is a commonly discussed problem [*Dixon et al.*, 1998]. Let us therefore augment our original one-fault network with an additional station designed to determine how much slip is partitioned between the original ("primary") fault, and a secondary fault suspected of activity. The velocity anywhere can be formed by superposition of velocities associated with each fault, where u_a and x_a are the slip rate and location of the primary fault, and u_b and x_b are of the secondary fault:

$$u(x) = \frac{u_a}{\pi}\arctan\frac{x - x_a}{D} + \frac{u_b}{\pi}\arctan\frac{x - x_b}{D} + \frac{u_a + u_b}{2} \tag{23}$$

(For simplicity, each fault has the same depth of locking D, but a result can be similarly derived for different values of D). From the single-fault case, the existing two stations in the far field are optimally located to determine the sum of slip rates. It is therefore convenient to re-parameterize the model in terms of the sum and difference of slip rates:

$$\begin{aligned} \mathbf{G}(x, u_-) = &\frac{u_+ + u_-}{2\pi}\arctan\frac{x - x_a}{D} \\ &+ \frac{u_+ - u_-}{2\pi}\arctan\frac{x - x_b}{D} + \frac{u_+}{2} \end{aligned} \tag{24}$$

To optimize the difference in slip ("slip partitioning"), let us first write

$$\det \mathbf{A} = \frac{\partial \mathbf{G}(x,u_-)}{\partial u_-} = \frac{1}{2\pi}\left(\arctan\frac{x-x_\mathrm{a}}{D} - \arctan\frac{x-x_\mathrm{b}}{D} \right) \quad (25)$$

and then find the station position that satisfies

$$0 = \frac{\partial}{\partial x}\det \mathbf{A} = \frac{1}{\left[(x-x_\mathrm{a})/D\right]^2+1} - \frac{1}{\left[(x-x_\mathrm{b})/D\right]^2+1} \quad (26)$$

The solution to this is

$$x = \frac{x_\mathrm{a}+x_\mathrm{b}}{2} = \frac{x_+}{2} \quad (27)$$

That is, the optimal location is precisely midway between the two faults. Note that this is independent of the magnitude of slip partitioning, hence this also optimizes testing of the null hypothesis (that the secondary fault is inactive). As an indicator of slip partitioning resolution:

$$\left|\det \mathbf{A}\right|_{\max} = \frac{1}{\pi}\arctan\frac{\left|x_\mathrm{a}-x_\mathrm{b}\right|}{2D} = \frac{1}{\pi}\arctan\frac{\left|x_-\right|}{2D} \quad (28)$$

Firstly, this implies that the resolution of slip partitioning (i.e., the formal error) is independent of its magnitude. Importantly, it also implies that resolution becomes fundamentally problematic for fault separations $|x_-|<2D$ (~30 km). At $|x_-|=D$ (~15 km) resolution is only 30% that of well-separated faults, with resolution dropping approximately linearly with decreasing fault separation. The basic principles discovered here with regard to resolution of slip partitioning can be extrapolated to multiple-fault systems. That is, stations should be placed midway between faults. A series of faults separated by less than the locking depth would fundamentally be difficult to resolve no matter how good the network design.

5. Conclusions

A new analytical method, generalized first order design is proposed for optimizing geodetic station locations for purposes of geophysical parameter estimation. The method, given by equation (13), finds the set of station locations that maximizes the determinant of the design matrix. This choice of optimization functional has appropriate qualities, including independence of units, and independence of parameter basis.

Analytical solutions to simple, idealized models might be used to seed numerical solution of more complex cases. Equation (14) specifically gives the optimization functional for SRIF, which can be implemented to incorporate multi-parameter stochastic modeling. Moreover, closed-form analysis might help develop insight towards efficient search algorithms. The method leads to exact analytical solutions for the case of the ideal, infinite transform fault. For example, to resolve simultaneously the depth of locking D and the location of the fault, optimal station locations are at $\pm D/\sqrt{3}$ from the a priori fault plane. Analysis of slip partitioning in a two-fault system shows that the resolution is optimized by including a station midway between faults; however resolution is fundamentally limited for faults separated by <2D (~30 km). Resolution is not found to depend on the magnitude of slip partitioning.

Investigators might intuitively design networks to see if details of the observed velocity field match expectations. This is a form of model validation, with different criteria (and therefore different resulting designs) than precision optimization. However, it can be treated as an optimization problem if formalized in terms of hypothesis testing. This would require prior selection of appropriate test parameters for a design strategy that optimizes model discrimination.

This paper has described a design methodology in an area that has practical and timely relevance, and helps to move geodesy's current emphasis on velocity precision towards geophysical model resolution. Further research should develop numerical application to more complicated systems. The method is quite general, and should find application to other areas, such as autonomous scheduling of spacecraft observations to optimize model resolution and hypothesis testing.

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