Self-consistency in reference frames, geocenter definition, and surface loading of the solid Earth

Geoffrey Blewitt[1]

Nevada Bureau of Mines and Geology and Seismological Laboratory, University of Nevada, Reno, Nevada, USA

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[1] Crustal motion can be described as a vector displacement field, which depends on both the physical deformation and the reference frame. Self-consistent descriptions of surface kinematics must account for the dynamic relationship between the Earth's surface and the frame origin at some defined center of the Earth, which is governed by the Earth's response to the degree-one spherical harmonic component of surface loads. Terrestrial reference frames are defined here as "isomorphic" if the computed surface displacements functionally accord with load Love number theory. Isomorphic frames are shown to move relative to each other along the direction of the load's center of mass. The following frames are isomorphic: center of mass of the solid Earth, center of mass of the entire Earth system, no-net translation of the surface, no-net horizontal translation of the surface, and no-net vertical translation of the surface. The theory predicts different degree-one load Love numbers and geocenter motion for specific isomorphic frames. Under a change in center of mass of surface load in any isomorphic frame, the total surface displacement field consists not only of a geocenter translation in inertial space, but must also be accompanied by surface deformation. Therefore estimation of geocenter displacement should account for this deformation. Even very long baseline interferometry (VLBI) is sensitive to geocenter displacement, as the accompanying deformation changes baseline lengths. The choice of specific isomorphic frame can facilitate scientific interpretation; the theory presented here clarifies how coordinate displacements and horizontal versus vertical motion are critically tied to this choice. *INDEX TERMS:* 1229 Geodesy and Gravity: Reference systems; 1247 Geodesy and Gravity: Terrestrial reference systems; 1213 Geodesy and Gravity: Earth's interior—dynamics (8115, 8120); 1223 Geodesy and Gravity: Ocean/Earth/atmosphere interactions (3339); *KEYWORDS:* reference frame, geocenter, surface loading, deformation, solid Earth, isomorphic

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1. Introduction

[2] Terrestrial reference frames play an integral role in the interpretation of global-scale, low-frequency phenomena in space-geodetic data [*Argus et al.*, 1999; *Chao et al.*, 1987; *Trupin et al.*, 1992; *Dong et al.*, 1997]. Indeed, the degree to which we can separate different processes in the Earth system using geodesy depends critically on our ability to realize accurate, stable reference frames, such as the International Earth Rotation Service (IERS) Terrestrial Reference Frame (ITRF) [*Altamimi et al.*, 2001]. For example, global sea level rise depends directly on the global balance of hydrologic processes and indirectly on crustal deformation due to various processes such as postglacial rebound and loading associated with the global hydrologic system itself. This situation requires a terrestrial reference frame defined consistently with the dynamics of loading so that crustal deformation can be accurately related to its effect on sea level. Global-scale processes typically have competing models that differ at the millimeter level, yet the differences at this level can have profound scientific implications and can lead to broad variation in socio-economic impact [*Bilham*, 1991].

[3] Reference frame realization and the separation of processes are coupled problems. For example, conservation of momentum demands that the center of mass of the Earth system (CM) is a kinematic fixed point, invariant to terrestrial dynamic processes. Realization of a frame centered on CM therefore plays a role in assessing the integration of various mass transport models and in defining "vertical displacement" for problems such as sea level rise. While CM can be realized through the well-modeled dynamics of Satellite Laser Ranging (SLR) satellites [*Watkins and Eanes*, 1997; *Chen et al.*, 1999], one problem is spatial and temporal aliasing, in that the sparsely observed SLR network is continuously deforming by various loading phenomena [*Van Dam et al.*, 2001]. The Global Positioning

[1]Also at School of Civil Engineering and Geosciences, University of Newcastle upon Tyne, UK.

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System (GPS) has the converse problem. GPS satellites are subject to complex nongravitational dynamics, which limits the dynamic interpretation of station motions in the CM frame in terms of global mass redistribution. However, GPS provides the level of high spatial and temporal resolution that is desirable for the study of solid Earth processes [*Blewitt*, 1993]. Therefore it is not sufficient to rely entirely on the CM frame for interpretation, but rather it is imperative to develop a theoretical framework that connects the CM frame to other, geometrically defined frames that are more generally accessible to techniques such as GPS and very long baseline interferometry (VLBI).

[4] Here I aim to develop self-consistency in dynamics models and terrestrial reference frame theory, as a contribution to improving geodetic practice and scientific interpretation. An important objective is to define terrestrial reference frames that are compatible with dynamic theory. Toward this objective I approach the general theoretical problem of describing vector displacements of the Earth's surface when the origin of the reference frame is dynamically related to surface loading. Degree-one deformation is fundamental to this problem [*Blewitt et al.*, 2001]. Another objective is to develop a methodology to classify frames, and so provide some indication as to which frame might be best suited to the research question at hand. For example, in one extreme case that requires explanation in this paper, *Blewitt et al.* [2001, p.2343] remarked that surface displacements associated with a change in the load's center of mass could be equally characterized as "either purely vertical or purely horizontal" depending on the selection of reference frame. Toward this objective I develop the concept of a general class of "isomorphic frames." Such frames provide dynamically consistent kinematic descriptions of Earth deformation despite predicting very different station coordinate time series. The apparent paradox will be explained in depth.

[5] The theoretical framework developed here should be useful (1) for comparing published results derived using different types of frames, (2) for selecting a frame appropriate to the problem at hand, (3) for self-consistent use of reference frames, data, and models, and (4) for the integration of space geodetic techniques. This work builds on the theory of geocenter motions of *Trupin et al.* [1992] and *Dong et al.* [1997], on reference frame concepts of *Argus* [1996], *Heki* [1996], and *Argus et al.* [1999], on the spectral loading theory of *Mitrovica et al.* [1994], and on degree-one deformation concepts of *Blewitt et al.* [2001]. Figure 1 presents a contextual overview of the work presented here, showing how these developments link to dynamic theory and to computed observation models.

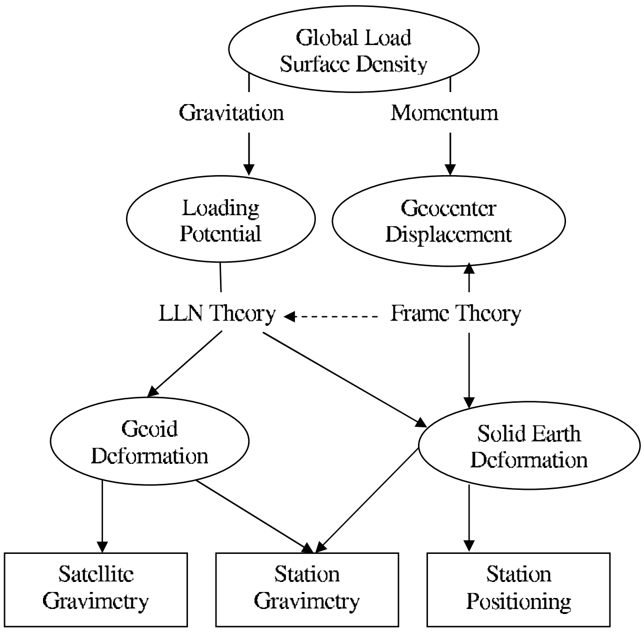

Figure 1. The basic elements of our analytical loading model, identifying how self-consistency between loading dynamics and the reference frame can be incorporated. Phenomena are in round boxes, measurement types are in rectangular boxes, and physical principles are attached to the connecting arrows. The arrows indicate the direction leading toward the computation of measurement models.

2. Solid Earth Loading Dynamics

2.1. Model Considerations

[6] Loading models have traditionally used Green's functions, as derived by *Farrell* [1972] and applied in various geodetic investigations [e.g., *Van Dam et al.*, 1994]. The Green's function approach is fundamentally based on load Love number theory, in which the Earth's deformation response is a function of the spherical harmonic components of the incremental gravitational potential created by the surface load. To study the interaction between loading dynamics and the terrestrial reference frame, it is convenient to use the spherical harmonic approach [*Lambeck*, 1988; *Mitrovica et al.*, 1994; *Grafarend et al.*, 1997] (therefore the conclusions would also apply to the use of Green's functions). The following "standard model" is based on a spherically symmetric, radially layered, elastic Earth statically loaded by a thin shell on the Earth's surface. *Farrell* [1972] used such a model to derive Green's functions that are now prevalent in atmospheric and hydrological loading models [e.g., *Van Dam et al.*, 2001]. The preliminary reference Earth model (PREM) [*Dziewonski and Anderson*, 1981] yields load Love numbers almost identical to those of Farrell [*Lambeck*, 1988; *Grafarend et al.*, 1997].

[7] Whereas it is in widespread use, the standard model might be improved by incorporating the Earth's ellipticity [*Wahr*, 1981], mantle heterogeneity [*Dziewonski et al.*, 1977; *Su et al.*, 1994; *Van Dam et al.*, 1994; *Plag et al.*, 1996], and Maxwell rheology [*Peltier*, 1974; *Lambeck*, 1988; *Mitrovica et al.*, 1994]. There is no consensus model to replace PREM yet; however, the general approach to reference frame considerations described here would be applicable to improved models (though it might require numerical rather than closed form analysis). The key point is it that a solid understanding on how to account for the choice of frame when interpreting data is needed for all models, especially those in routine use, and that a lack of this understanding might lead to errors of interpretation.

2.2. Equilibrium Tide

[8] It is analytically convenient to decompose the Earth system as a spherical, solid Earth of radius R_E and surface

mass that is free to redistribute in a thin surface layer ($\ll R_E$) of surface density $\sigma(\Omega)$ which is a function of geographical position Ω (latitude φ, longitude λ). Let us express the total redistributed load as a spherical harmonic expansion:

$$\sigma(\Omega) = \sum_{n=1}^{\infty} \sum_{m=0}^{n} \sum_{\Phi=C}^{S} \sigma_{nm}^{\Phi} Y_{nm}^{\Phi}(\Omega), \tag{1}$$

where $Y_{nm}^{\Phi}(\Omega)$ are defined in terms of associated Legendre polynomials $Y_{nm}^{C} = P_{nm}$ (sin φ) cos $m\lambda$, $Y_{nm}^{S} = P_{nm}$ (sin φ) sin $m\lambda$. The summation begins at degree $n = 1$ assuming that mass is conserved in the Earth system. It is this initial degree-one term that is used in section 3 to address the reference frame problem.

[9] It can be shown [e.g., *Bomford*, 1980, p.408] that, for a rigid Earth, such a thin-shell model produces the following incremental gravitational potential at the Earth's surface, which we call the "load potential":

$$V(\Omega) = \sum_n V_n(\Omega) = \frac{4\pi R_E^3 g}{M_E} \sum_n \sum_m \sum_{\Phi} \frac{\sigma_{nm}^{\Phi} Y_{nm}^{\Phi}(\Omega)}{(2n+1)}, \tag{2}$$

where g is acceleration due to gravity at the Earth's surface and M_E is the mass of the Earth. This load potential results in a displacement of the geoid, called the "equilibrium tide." As shall be addressed later, the load deforms the solid Earth, and in doing so creates an additional potential.

2.3. Surface Deformation

[10] According to load Love number theory, solutions for surface displacements $\Delta s_h(\Omega)$ in the local height direction and $\Delta s_l(\Omega)$ in any lateral direction specified by unit vector $\hat{\mathbf{l}}(\Omega)$ are given [*Lambeck*, 1988] by

$$\begin{aligned} \Delta s_h(\Omega) &= \sum_n h_n' V_n(\Omega)/g \\ \Delta s_l(\Omega) &= \sum_n l_n' \hat{\mathbf{l}}(\Omega). \nabla V_n(\Omega)/g, \end{aligned} \tag{3}$$

and the additional potential caused by the resulting deformation is

$$\Delta V(\Omega) = \sum_n k_n' V_n(\Omega), \tag{4}$$

where h_n', l_n', and k_n' are degree-n load Love numbers, with the prime distinguishing Love numbers used in loading theory from those used in tidal theory. The surface gradient operator is defined $\nabla = \hat{\varphi}\partial_{\phi} + \hat{\lambda}(1/\cos\phi)\partial_{\lambda}$, where $\hat{\varphi}$ and $\hat{\lambda}$ are unit vectors pointing northward and eastward, respectively.

[11] The net loading potential (load plus additional potential) relative to an Eulerian observer (the "space potential" as observed on a geocentric reference surface) is

$$U(\Omega) = V(\Omega) + \Delta V(\Omega) = \sum_n \left(1 + k_n'\right) V_n(\Omega). \tag{5}$$

The net loading potential relative to a Lagrangean observer (the "body potential" as observed on the deforming Earth's surface) must also account for the lowering of Earth's surface due to loading. From equations (5) and (3), the body potential is

$$U'(\Omega) = U(\Omega) - g\Delta s_h(\Omega) = \sum_n \left(1 + k_n' - h_n'\right) V_n(\Omega). \tag{6}$$

Therefore the "space" and "body" combinations of load Love number, $(1 + k_n')$ and $(1 + k_n' - h_n')$ are relevant to computing gravity acting on Earth-orbiting satellites and Earth-fixed instruments, respectively.

[12] Solutions for surface deformations of the thin-shell loading model are found by substituting equation (2) into equations (3) and (5):

$$\begin{aligned} \Delta s_h(\Omega) &= \frac{4\pi R_E^3}{M_E} \sum_n \sum_m \sum_{\Phi} h_n' \frac{\sigma_{nm}^{\Phi} Y_{nm}^{\Phi}(\Omega)}{2n+1}, \\ \Delta s_l(\Omega) &= \frac{4\pi R_E^3}{M_E} \sum_n \sum_m \sum_{\Phi} l_n' \frac{\sigma_{nm}^{\Phi} \hat{\mathbf{l}}. \nabla Y_{nm}^{\Phi}(\Omega)}{2n+1}, \\ U(\Omega) &= \frac{4\pi R_E^3}{M_E} \sum_n \sum_m \sum_{\Phi} \left(1 + k_n'\right) \frac{\sigma_{nm}^{\Phi} Y_{nm}^{\Phi}(\Omega)}{2n+1}. \end{aligned} \tag{7}$$

3. Linking Dynamics to the Reference Frame

[13] Equation (7) tells us the difference in position (and potential) before and after imposing a load surface density distribution parameterized in terms of spherical harmonic coefficients. It will now be shown that the degree-one contributions depend on the choice of reference frame, specifically how the origin moves relative to the deforming Earth. A fundamental problem that will now be addressed is that a constant translation can be applied to equation (7) without changing its form. Note that a constant translation is a special case of a degree-one surface displacement field, which alone would describe the degree-one loading response only if the Earth were perfectly rigid. The degree-one deformation field for a nonrigid Earth can be described as a combination of deformation field plus a translation.

3.1. Degree-One Load Deformations

[14] For reference frame analysis it is useful to reformulate the degree-one potential in terms of the center of mass of the load. From equation (2) we have

$$V_1(\Omega) = \frac{4\pi\alpha^3 g}{3M_E} \left(\sigma_{11}^C Y_{11}^C(\Omega) + \sigma_{11}^S Y_{11}^S(\Omega) + \sigma_{10}^C Y_{10}^C(\Omega)\right), \tag{8}$$

where $Y_{11}^C(\Omega) = \cos\varphi \cos\lambda$, $Y_{11}^S(\Omega) = \cos\varphi \sin\lambda$, $Y_{10}^C(\Omega) = \sin\varphi$. *Blewitt et al.* [2001] encapsulate the relevant properties of the load defining the load moment vector

$$\mathbf{m} = M_L \Delta\bar{\mathbf{r}}_L, \tag{9}$$

where $\Delta\bar{\mathbf{r}}_L$ is the change in the load's center of mass (relative to the solid Earth center of mass) and M_L is the mass of that portion of the load going into the calculation of $\Delta\bar{\mathbf{r}}_L$. Typical magnitudes for the case of seasonal groundwater variations are $\Delta\bar{r}_L \sim 10^6$ m and $M_L \sim 10^{16}$ kg. Therefore, unlike the case for the actual deformations ($\sim$10 mm), $\mathbf{m}$ is not sensitive to the specific choice of reference frame, as long as it is approximately geocentric. Note that **m** has the useful property that it is independent of whether we reckon stationary mass as part of the "redistributed" load.

[15] As the load is restricted to move on a spherical surface of radius R_E, equation (9) is equivalent to

$$\begin{aligned}\mathbf{m} &= \iint_\Omega R_E \hat{\mathbf{h}}(\Omega)\sigma(\Omega)R_E^2 d\Omega \\ &= R_E^3 \iint_\Omega \left(Y_{11}^C(\Omega)\hat{\mathbf{x}} + Y_{11}^S(\Omega)\hat{\mathbf{y}} + Y_{10}^C(\Omega)\hat{\mathbf{z}}\right)\sigma(\Omega)d\Omega, \qquad (10)\end{aligned}$$

where $d\Omega = \sin\lambda \, d\lambda d\varphi$ is an element of solid angle, $\hat{\mathbf{h}}(\Omega)$ is a unit vector pointing locally upward, and we use the identities $\hat{\mathbf{h}}.\hat{\mathbf{x}} = Y_{11}^C$, $\hat{\mathbf{h}}.\hat{\mathbf{y}} = Y_{11}^S$, and $\hat{\mathbf{h}}.\hat{\mathbf{z}} = Y_{10}^C$. Substitution of equation (1) into equation (10) and performing the integration shows explicitly a one-to-one correspondence between load moment components and degree-one coefficients of surface density:

$$\left(m_x, m_y, m_z\right) = \frac{4\pi R_E^3}{3}\left(\sigma_{11}^C, \sigma_{11}^S, \sigma_{10}^C\right). \qquad (11)$$

[16] Hence, equation (8) can be reformulated as

$$V_1(\Omega) = \frac{g}{M_E}\left(m_x Y_{11}^C(\Omega) + m_y Y_{11}^S(\Omega) + m_z Y_{10}^C(\Omega)\right) = g\hat{\mathbf{h}}(\Omega).\,\mathbf{m}/M_E. \qquad (12)$$

[17] *Blewitt et al.* [2001] derived the same equation less rigorously as a "first-order" calculation, arguing that the total momentum of the Earth-load system is conserved and that thus the solid Earth must experience a tide-raising potential as it is displaced relative to CM through its own degree-zero gravity field. The derivation here proves the postulate of *Blewitt et al.* [2001] that equation (12) corresponds to the degree-one component of the load potential. Substituting the vector form of equation (12) into equations (3) and (5) gives us the following alternative form for the degree-one deformations (where, to reduce clutter, the functional dependence on Ω is now implicit):

$$\begin{aligned}\Delta s_h &= h_1' V_1/g = h_1'\hat{\mathbf{h}}.\,\mathbf{m}/M_E, \\ \Delta s_l &= l_1'\hat{\mathbf{l}}.\nabla V_1/g = l_1'\hat{\mathbf{l}}.\,\mathbf{m}/M_E, \qquad (13) \\ U_1 &= \left(1 + k_1'\right)V_1 = \left(1 + k_1'\right)g\hat{\mathbf{h}}.\,\mathbf{m}/M_E.\end{aligned}$$

The derivation of Δs_l in this equation uses an identity in vector analysis, proved in Appendix A. Next, I show that load Love numbers depend on how the origin of the reference frame translates relative to the Earth's surface when the load is redistributed. Specifically, the form of equation (13) is independent under translations along the direction of **m**.

3.2. Isomorphic Frame Transformations

[18] Let us assume we have a specified Earth model reference frame in which load Love numbers have been derived. How should we transform the load Love numbers into different reference frames? In the context of loading theory, I define a reference frame as "isomorphic" if the degree-one deformations take the form of equation (13). What distinguishes specific isomorphic frames is the set of load Love numbers; but the functional dependence on the load moment is invariant.

[19] I now demonstrate that isomorphic frames have the property that, under redistribution of the load characterized by load moment **m**, the origins translate relative to each along the axis of **m**. First let us postulate this and parameterize the translation of frame B with respect to frame A by

$$[\mathbf{t}_B]_A = [\alpha_B]_A \mathbf{m}/M_E, \qquad (14)$$

where the "isomorphic parameter" α depends on the conceptual definition of the reference frame origin and M_E ensures that α is dimensionless. The subscripted square brackets indicate to which frame the parameter refers. In the new frame B the deformations appear to be

$$\begin{aligned}[\Delta s_h]_B &= \left[\Delta s_h - \hat{\mathbf{h}}.\mathbf{t}_B\right]_A, \\ [\Delta s_l]_B &= \left[\Delta s_l - \hat{\mathbf{l}}.\mathbf{t}_B\right]_A, \qquad (15) \\ [U]_B &= \left[U - g\hat{\mathbf{h}}.\mathbf{t}_B\right]_A.\end{aligned}$$

Inserting equations (13) and (14) into equation (15) and rearranging, we find

$$\begin{aligned}[\Delta s_h]_B &= \left[h_1' - \alpha_B\right]_A \hat{\mathbf{h}}.\,\mathbf{m}/M_E, \\ [\Delta s_l]_B &= \left[l_1' - \alpha_B\right]_A \hat{\mathbf{l}}.\,\mathbf{m}/M_E, \qquad (16) \\ [U]_B &= \left[1 + k_1' - \alpha_B\right]_A g\hat{\mathbf{h}}.\,\mathbf{m}/M_E.\end{aligned}$$

Therefore, equation (16) has the same form as equation (13), where new degree-one load Love numbers are defined by

$$\begin{aligned}\left[h_1'\right]_B &= \left[h_1' - \alpha_B\right]_A, \\ \left[l_1'\right]_B &= \left[l_1' - \alpha_B\right]_A, \qquad (17) \\ \left[1 + k_1'\right]_B &= \left[1 + k_1' - \alpha_B\right]_A.\end{aligned}$$

Note that a translation of the frame origin only affects the degree-one load Love numbers; therefore, equation (17) does not apply to higher degrees. This can be understood if we consider a rigid body translation as a special case of degree-one deformation and consider that higher degree deformations are orthogonal to degree-one; hence, the degree-one components account for the translation entirely.

[20] It remains for us to calculate the isomorphic parameter α for the several types of reference frames used in global deformation research and then use equation (17) to compute the new load Love numbers. Once this is achieved, then equation (7) can be used to compute deformations in the new frame, given a model of the load. Alternatively, inversion using geodetic data on surface displacements might be used to estimate the load surface density distribution σ_{nm}^Φ, for which the load Love numbers must be transformed to the geodetic frame of choice [*Blewitt et al.*, 2001].

3.3. Reference Frame Properties of Degree-One Deformations

[21] Equation (16) can be used to illustrate properties of degree-one deformations that might be counterintuitive.

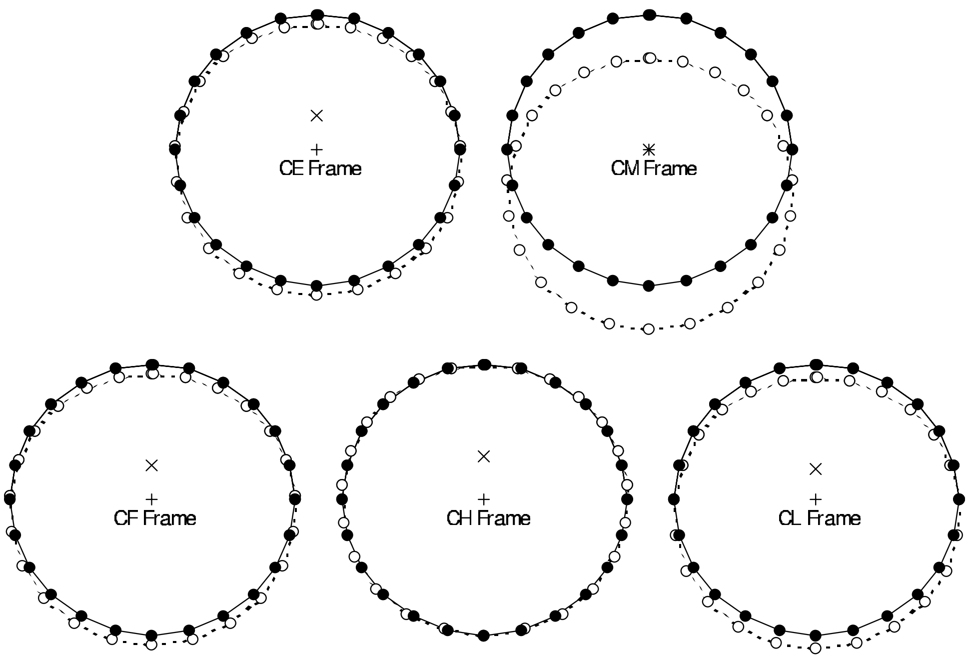

Figure 2. Degree-one deformations in various frames, generated by equation (16). A cross section of the Earth is shown where the load moment points along the Earth's axis toward the North Pole (top of the figure). Filled circles show stations on the Earth's surface (solid line) prior to deformation. Open circles show stations on the Earth's surface (dashed line) after deformation for each indicated frame. By definition, each frame origin (+) remains stationary during mass redistribution and subsequent deformation. Within each frame, CM is displaced from the origin to the location shown (×).

Figure 2 was computer-generated assuming specific values of α in equation (16), starting with Farrell's load Love numbers given by equation (18). Figure 2 illustrates an exaggerated degree-one deformation field as seen in the frames variously defined in the next section of this paper, where the load moment points along the Earth's axis toward the North Pole (top of the figure). Starting with stations (filled circles) on the Earth's surface (solid line), the Earth's surface deforms (dashed line) and the stations move to their final positions (open circles).

[22] The following key characteristics of degree-one deformation are noted: (1) the Earth's surface remains a perfect sphere that is displaced with respect to the original sphere; (2) the stations move closer together in the Northern Hemisphere and further apart in the Southern Hemisphere; (3) the shape of the final polyhedron of stations is frame invariant; (4) the difference in sets of final station coordinates in each frame is given by a common translation (often called the "geocenter"); and therefore (5) in all frames the difference between initial and final station coordinates can be described by a frame-independent deformation plus a frame-dependent translation.

[23] In addition to these characteristics, note that a new frame can always be found for which any of the three load Love numbers are zero. This implies that any degree-one deformation field can be transformed into a frame in which the displacements appear either purely vertical (the CL frame in Figure 2) or purely horizontal (CH frame). Both points of view are equally valid. There is no fundamental distinction between the concepts of vertical motion versus horizontal motion, as they depend entirely on the choice of reference frame. Therefore, great care must be exercised when interpreting global-scale deformations involving global-scale redistribution of surface load, especially when questions use the terms vertical and horizontal.

[24] Another unusual property is that if we chose a frame in which the surface motion is entirely horizontal (CH frame), then station motions actually peak at a maximal distance (90°) away from the point of maximum/minimum loads. In such a frame, peak motions at the equator correspond to peak loads at the poles. This illustrates severe limitations in nonglobal models of loading that use linear regression of modeled displacement to local pressure, such as recommended in the IERS conventions [*McCarthy*, 1996]. From a terrestrial reference frame perspective, local models are not useful and are likely to be misleading (perhaps improving scatter, but not accuracy) for purposes of separation of global-scale Earth processes.

4. Linking Frame Theory to Frame Practice

[25] There are several conceptually different types of reference frame that have been used in theory and practice [*Heflin et al.*, 1992; *Watkins et al.*, 1994; *Van Dam et al.*, 1994; *Argus*, 1996; *Heki*, 1996; *Dong et al.*, 1997; *Ma and Ryan*, 1997; *Ray*, 1998; *Argus et al.*, 1999; *Chen et al.*, 1999; *Gerasimenko and Kato*, 2000; *Altamimi et al.*, 2001]. Love number transformations given by equation (17) are only applicable for the class of isomorphic reference frames which, under a redistribution of load, move relative to each

other along the axis of the load moment, **m**. Fortunately, many conceptual reference frames used in practice satisfy this criterion, including frames for which the origin is defined in terms of center of mass or some geometric center of figure. We shall now consider specific conceptual isomorphic frames, which allow us to calculate then how the origins and the degree-one load Love numbers transform between them.

4.1. Center of Mass of the Solid Earth (CE)

[26] Let us first consider a reference frame fixed to the center of mass of the solid Earth, CE. As shall be discussed later, CE does change its trajectory in inertial space when surface mass is redistributed. However, once the mass has been redistributed to its final configuration, any resulting deformation of the solid Earth cannot itself change the solid Earth's center of mass. Hence, the degree-one space potential of equation (5) remains invariant to deformation in the CE frame, so $[k_1']_{\mathrm{CE}} = 0$ Therefore, the CE frame is a natural frame to compute the dynamics of solid Earth deformation and to model load Love numbers. *Farrell* [1972] specifies that in the CE frame

$$\begin{aligned} \left[h_1'\right]_{\mathrm{CE}} &= -0.290, \\ \left[l_1'\right]_{\mathrm{CE}} &= 0.113, \\ \left[1+k_1'\right]_{\mathrm{CE}} &= 1. \end{aligned} \tag{18}$$

[27] A major disadvantage of the CE frame, however, is that it is not directly accessible to observation. As pointed out by *Dong et al.* [1997], the CE frame closely approximates the CF frame (center of surface figure, defined below). However, to rigorously compare observation to model requires a transformation of coordinates from the model CE frame into the appropriate observational reference frame. The alternative given in this paper is to apply equation (17) to transform the load Love numbers to an appropriate observational frame.

4.2. Center of Mass of the Earth System (CM)

[28] The center of mass of the entire Earth system includes the solid Earth and surface load. CM is stationary with respect to satellite orbits in inertial space and is therefore an appropriate frame for modeling SLR measurements. In practice, for GPS, the procedure of simultaneously estimating satellite orbits and fiducial-free station positions [*Heflin et al.*, 1992] naturally produces station coordinates in the CM frame. While the internal geometry of a fiducial-free network can be very precise, the external solution is typically an order of magnitude less precise due to sensitivity of a global translation to nongravitational forces acting on the satellites [*Vigue et al.*, 1992]. For this reason CM coordinate solutions are often transformed so that there is no-net translation with respect to some previously established CM frame, which may require site ties (local surveys) to collocated stations [*Blewitt et al.*, 1992; *Altamimi et al.*, 2001]. VLBI site velocities have been placed in the CM frame by applying a global velocity that minimizes velocity differences with SLR at sites collocated by the two techniques [*Watkins et al.*, 1994]. In all these cases that involve frame transformations, however, it must be recognized that the degree to which the solution is useful as a CM solution depends on the accuracy and the temporal resolution of the frame to which it is tied. For example, ITRF2000 [*Altamimi et al.*, 2001] might be useful as a CM frame to study secular deformation but not for seasonal loading.

[29] To conserve momentum, redistribution of surface mass displaces CE relative to CM by the geocenter displacement vector $[\mathbf{t}_{\mathrm{CE}}]_{\mathrm{CM}}$, which satisfies [*Trupin et al.*, 1992; *Dong et al.*, 1997; *Blewitt et al.*, 2001]

$$[\mathbf{t}_{\mathrm{CE}}]_{\mathrm{CM}} = -M_L \Delta\bar{\mathbf{r}}_L / M_E = -m/M_E = -[\mathbf{t}_{\mathrm{CM}}]_{\mathrm{CE}}, \tag{19}$$

which uses the definition of **m** from equation (9). From equation (14), therefore, we have $[\alpha_{\mathrm{CM}}]_{\mathrm{CE}} = 1$. Inserting this into equation (17) and using Farrell's numbers of equation (18), we find

$$\begin{aligned} \left[h_1'\right]_{\mathrm{CM}} &= \left[h_1'\right]_{\mathrm{CE}} - 1 = -1.290, \\ \left[l_1'\right]_{\mathrm{CM}} &= \left[l_1'\right]_{\mathrm{CE}} - 1 = -0.887, \\ \left[1+k_1'\right]_{\mathrm{CE}} &= \left[1+k_1'\right]_{\mathrm{CE}} - 1 = 0. \end{aligned} \tag{20}$$

[30] Note that the displacement field does not look like a rigid body translation ($h_1' = l_1' = -1$), as has until now been an unchallenged assumption in satellite geodetic analysis of the geocenter. From equation (20), station coordinate displacements in the CM frame due to degree-one loading will be 29% larger in the height component and 11% smaller in lateral components than the displacements predicted by a rigid body translation model. Even more significant is the effect on relative position. For example, for a load moment pointing along the z axis, the relative vector between a station at the pole and a station on the equator changes (along the z axis) by 40% of the rigid body translation vector (also along the z axis).

[31] Although the total degree-one space potential has zero effect on the satellite orbits in the CM frame, not accounting for the stations displacements will affect the determination of the satellite orbits, which will further feedback into error in the determination of CM relative to the stations in a complicated way. Therefore, methods that estimate degree-one perturbations to satellite orbits (as practiced in SLR) but do not allow the station network to deform are inherently inconsistent. For example, *Chen et al.* [1999] ignore the deformation, assuming that this effect is "very small (about 2%)," a number which correctly relates to a frame origin effect [*Dong et al.*, 1997] but which does not account for relative station displacements (up to 40% of the CE translation) and how this error propagates into the estimated orbits.

4.3. Center of Surface Figure (CF)

[32] Let us define the center of surface figure, CF, following the notation of *Dong et al.* [1997], who in turn followed *Trupin et al.*'s [1992] definition of "center of surface." The CF frame is defined geometrically as though the Earth's surface were covered by a uniform, infinitely dense array of points and the motions of these points are taken into account. In practice, this can be realized by appropriately averaging over a sufficiently dense global distribution of geodetic stations, which presents no problem in the case of GPS. In theory, the origin of the CF frame is

such that the surface integral of the vector displacement field is zero. This corresponds to no-net translation projected along any axis.

[33] It is important to recognize that rigid plate tectonics, with boundaries that create and destroy surface, will generally produce a net translation of CF in the CM frame. The average tectonic motion of the Earth's surface is actually more northward than southward, which implies a net positive z component of secular translation of CF with respect to the fixed point of plate rotation (modeled at the center of a sphere). In contrast, physical arguments suggest that CM cannot have significant secular motion with respect to the center of the sphere [*Argus et al.*, 1999, p. 29,080].

[34] It is therefore more appropriate to apply a no-net translation condition to the residual station velocities (observed minus plate rotation model), but strictly speaking, this is not a CF frame. However, it is essential to calibrate for tectonics when investigating loading. A residual CF frame to study loading can be defined by applying a no-net translation at every epoch with respect to a secular CM frame [*Davies and Blewitt*, 2000]. Such a procedure has been used to produce suitable CF frames for investigating nonsecular loading [*Van Dam et al.*, 1994, 2001; *Blewitt et al.*, 2001].

[35] In practice, the CF frame is well suited for techniques where CM cannot be accurately realized. It is a natural frame of choice for GPS, for which stochastic variation in radiation pressure on the satellites is a limiting error source. It is also a natural frame for VLBI using quasar sources, with which no satellite dynamics are involved.

[36] To compute coordinate displacements in the CF frame, we can integrate the vector surface displacements in equation (16) and then solve for α such that the integral is zero.

$$\begin{aligned}
&\iint_\Omega [\Delta\mathbf{s}]_{CF}\, d\Omega = 0,\\
&\iint_\Omega \Big([l'_1]_{CF}(\mathbf{m}-\hat{\mathbf{h}}m_h) + [h'_1]_{CF}\hat{\mathbf{h}}m_h\Big)d\Omega = 0,\\
&\iint_\Omega \Big([l'_1-\alpha_{CF}]_{CE}(\mathbf{m}-\hat{\mathbf{h}}m_h) + [h'_1-\alpha_{CF}]_{CE}\hat{\mathbf{h}}m_h\Big)d\Omega = 0,\\
&\iint_\Omega \Big([l'_1-\alpha_{CF}]_{CE}\mathbf{m} + [h'_1-l'_1]_{CE}\hat{\mathbf{h}}(\hat{\mathbf{h}}.\mathbf{m})\Big)d\Omega = 0.
\end{aligned} \tag{21}$$

Given that $\mathbf{m}$ is a global constant not subject to surface integration, we are free to take the dot product of the integrand with $\mathbf{m}$ (to turn this into a scalar equation):

$$\begin{aligned}
&\iint_\Omega \Big([l'_1-\alpha_{CF}]_{CE}m^2 + [h'_1-l'_1]_{CE}(\hat{\mathbf{h}}.\mathbf{m})^2\Big)d\Omega = 0,\\
&\iint_\Omega \Big([l'_1-\alpha_{CF}]_{CE} + [h'_1-l'_1]_{CE}Y'^2_{10}\Big)d\Omega = 0,\\
&4\pi[l'_1-\alpha_{CF}]_{CE} + \frac{4\pi}{3}[h'_1-l'_1]_{CE} = 0,\\
&[\alpha_{CF}]_{CE} = \frac{1}{3}[h'_1+2l'_1]_{CE},
\end{aligned} \tag{22}$$

where Y'_{10} is a degree-one zonal harmonic along the axis of $\mathbf{m}$. This result is consistent with that of *Trupin et al.* [1992] (later used by *Dong et al.* [1997]), who integrated the complex spherical harmonic form of loading deformation to compute the translation between CF and CE. The difference here is that I identify the parameter α_{CF} with a load Love number transformation, equation (17). Inserting this into equation (17) and applying equation (18) gives

$$\begin{aligned}
[h'_1]_{CF} &= \frac{2}{3}[h'_1-l'_1]_{CE} = -0.268,\\
[l'_1]_{CF} &= -\frac{1}{3}[h'_1-l'_1]_{CE} = 0.134,\\
[1+k'_1]_{CF} &= \left[1-\frac{1}{3}h'_1-\frac{2}{3}l'_1\right]_{CE} = 1.021.
\end{aligned} \tag{23}$$

[37] Comparing equations (23) and (18), we therefore conclude that load Love numbers in the CE frame commonly used for dynamic modeling are not identical to those in the CF frame commonly used in geodetic practice. Fortunately the difference is small. *Dong et al.* [1997] indicated that if the Earth were homogeneous, CF and CE would be equivalent. For Farrell's model, the trajectories of CF and CE agree to within 2% of their magnitudes. Therefore, an advantage of CF is that it approximates the center of mass of the solid Earth.

4.4. Center of Surface Lateral Figure (CL)

[38] In theory, the center of lateral figure (CL) frame is such that the surface integral of the horizontal vector displacement field is zero. In practice this type of constraint has been applied to residuals of observed tectonic motions minus modeled plate tectonic motions [*Heki*, 1996; *Ma and Ryan*, 1997]. Such a method serves to realize a frame consistent with plate tectonics but does not realize a true CL frame because plate tectonics produces a nonzero net horizontal vector displacement (again, due to surface creation and destruction). However, such a procedure can effectively produce a "residual CL frame" for purposes of investigating loading dynamics. *Lavallée* [2000] defines a type of residual CL frame that minimizes the rate of spherical distance between stations on rigid plate interiors, without use of an a priori plate rotation model.

[39] The CL reference frame can be useful if vertical displacements are much less accurately determined (or modeled) than are lateral displacements. The CL frame may be applicable to radio techniques such as GPS and VLBI, where vertical accuracy can be limited by errors in atmospheric refractivity due to stochastic water vapor variations.

[40] By inspection of equation (17), choosing $\alpha_{CL} = l'_1$ defines a reference frame in which there is no lateral motion caused by the degree-one deformations. Since only the degree-one deformations can produce average translations of the Earth's surface, we can conclude that this defines the CL frame.

$$\begin{aligned}
[h'_1]_{CL} &= [h'_1-l'_1]_{CE} = -0.403,\\
[l'_1]_{CL} &= [l'_1-l'_1]_{CE} = 0,\\
[k'_1+1]_{CL} &= [1-l'_1]_{CE} = 0.887.
\end{aligned} \tag{24}$$

[41] Note that, in comparison with the CE frame, commonly used for dynamic modeling, degree-one vertical

deformations in the CL frame are enhanced by ∼30%. This type of effect might occur if stations are constrained to move with constant horizontal velocities.

4.5. Center of Surface Height Figure (CH)

[42] The no-net height displacement frame (CH) is a center of figure frame such that the vector average of height displacements over the Earth's surface is zero. This is perhaps the most intuitive geometrical frame, as CH is centered on the spherical shape of the Earth's surface, which remains spherical under degree-one deformation. CH is a natural frame for models of vertical motion only, where the globally averaged vertical motion is believed to be better modeled than is horizontal motion or where geological plate rotation models are suspect [*Gerasimenko and Kato*, 2000]. In practice, this type of constraint has been applied to velocity residuals of observed rebound minus modeled post-glacial rebound [*Argus*, 1996], thus realizing a "residual (rebound-adjusted) CH frame." Vertical velocities of several stations in global solutions have been constrained to zero (C. Ma et al., Goddard Space Flight Center (GSFC) VLBI solutions GLB886a; GLB907, 1993, 1994, available at http://lupus.gsfc.nasa.gov/global/glb.html), thus effectively realizing a type of CH frame (though not in a true global-average sense).

[43] By inspection of equation (17), choosing $\alpha_{CH} = h_1'$ defines a reference frame in which there is no height motion caused by the degree-one deformations. Since only the degree-one deformations can produce average translations of the Earth's surface, we can conclude that this defines the CH frame.

$$\begin{aligned} [h_1']_{CH} &= [h_1' - h_1']_{CE} = 0, \\ [l_1']_{CH} &= [l_1' - h_1']_{CE} = 0.403, \\ [1 + k_1']_{CH} &= [1 - h_1']_{CE} = 1.290. \end{aligned} \qquad (25)$$

[44] Note that, in comparison with the CE frame commonly used for dynamic modeling, degree-one lateral deformations are enhanced by a factor of ∼3 and are now even larger than the height deformations in the CE frame. This type of effect might occur if stations are constrained to have no height motion. It is clear that horizontal motion can entirely absorb degree-one deformations; therefore, the danger of constraining the heights of stations around the globe is that it can greatly magnify loading variations in the horizontal. Also note that the total potential at radius R_E in the CH frame is precisely equal to the total body potential at the Earth's surface given by equation (6), which of course it should, given that in the CH frame, the surface remains at radius R_E.

4.6. General Isomorphic Frames

[45] In principle, other isomorphic frames can be defined by constraining any of the degree-one load Love numbers, or linear combinations thereof, to any value. Isomorphic frames can also be generated by taking linear combinations of parameter α associated with other isomorphic frames. In these cases it is straightforward to apply the analysis method here to derive new load Love numbers. Whether or not combination frames are practical depends on the problem at hand. For example, *Argus et al.* [1999] produce a combination of a rebound-adjusted CH frame together with a residual CL frame (using rebound-adjusted velocities with respect to estimated rigid plate motions).

[46] Frames that constrain the coordinates of individual stations will generally not satisfy equation (16) and so are not isomorphic. Even minimal constraint frames will generally not be isomorphic if constrained station coordinates do not incorporate an accurate loading displacement model. Frames that are not isomorphic should be avoided, as interpretation would not be straightforward. Fiducial-free analyses, however, allow for nonrigid body deformation. Coordinate time series can be constructed for station displacements relative to linear combinations of station displacements that (ideally) would satisfy the definition of a convenient isomorphic frame [*Blewitt et al.*, 2001]. Interpretation could then proceed on the understanding that we must use the degree-one load Love number appropriate to the chosen reference frame.

4.7. Geocenter Variations

[47] A change in the center of mass of surface loads induces a detectable translation of the solid Earth relative to the center of satellite orbits [*Watkins and Eanes*, 1997; *Chen et al.*, 1999]. Observations of this phenomenon help to constrain models involving global redistribution of mass, especially the water cycle and how it might be affected by global climate change [*Chao et al.*, 1987; *Trupin et al.*, 1992; *Dong et al.*, 1997]. The physical principle behind the "geocenter" phenomenon is conservation of momentum, and the only property of the solid Earth that is relevant is its mass.

[48] However, it is important to recognize that the real distance-changing deformation associated with change in the center of mass of the surface load is not a negligible effect, despite the (perhaps misleading) fact that the CF frame closely approximates the CE frame. As seen previously, the vectors between individual stations vary by up to 40% of the geocenter trajectory. Hence, a cause for concern is that previous methods determining the geocenter trajectory might be biased unless they also account for the sizable distance-changing deformation associated with change in load center of mass.

[49] The term "geocenter" generically refers to the translation of the origin between terrestrial reference frames; however, its definition is not universally accepted. For example, *Chen et al.* [1999] define the geocenter as the translation of CM with respect to CE; *Dong et al.* [1997] define the geocenter as the translation of CF with respect to CM. *Argus* [1996] defines the geocenter as the translation of rebound-adjusted CH (with the intent of aligning to the rebound model's CE frame) with respect to CM. *Argus et al.* [1999] define the geocenter generally as any geometrical center of figure with respect to CM.

[50] Table 1 summarizes the results of our derivations for load Love numbers in various isomorphic frames appropriate for geodetic analysis, along with the translations of origin. I choose to express isomorphic parameters relative to CE because that is an appropriate frame for solid Earth dynamics. However, I choose the convention of *Argus et al.* [1999], expressing translations with respect to CM, because CM is unperturbed by load redistribution and is the natural reference frame origin for modeling satellite dynamics.

Table 1. Degree-One Load Love Numbers and Relative Translations for Various Isomorphic Frames

Description of Frame B	Isomorphic Parameter Relative to CE $[\alpha_B]_{CE}$	Load Love Number: Height $[h'_1]_B$	Lateral $[l'_1]_B$	Space Potential $[1+k'_1]_B$	Body Potential $[1+k'_1-h'_1]_B$	Geocenter Relative to CM $[\mathbf{t}_B]_{CM}$
Center of mass of solid Earth, CE	0	−0.290	0.113	1	1.290[a]	$-\mathbf{m}/M_E$
Center of mass of entire Earth, CM	1	−1.290	−0.887	0	1.290[a]	0
Center of surface figure, CF	$\frac{1}{3}[h'_1 + 2l'_1]_{CE} = -0.021$	−0.269	0.134	1.021	1.290[a]	$-1.021(\mathbf{m}/M_E)$
Center of lateral surface figure, CL	$[l'_1]_{CE} = 0.113$	−0.403	0	0.887	1.290[a]	$-0.887(\mathbf{m}/M_E)$
Center of height surface figure, CH	$[h'_1]_{CE} = -0.290$	0	0.403	1.290[a]	1.290[a]	$-1.290(\mathbf{m}/M_E)$

[a]This provides a consistency check, knowing that the body potential calculated at a point moving with the Earth's surface should not depend on the choice of origin.

[51] The translation of origin of an isomorphic frame B as viewed in the CM frame is related to the load moment by the following formula:

$$[\mathbf{t}_B]_{CM} = [\mathbf{t}_B - \mathbf{t}_{CM}]_{CE} = [\alpha_B - \alpha_{CM}]_{CE}\mathbf{m}/M_E = \left([\alpha_B]_{CE} - 1\right)\mathbf{m}/M_E. \tag{26}$$

Equation (26) was used to generate the location of CM in the various frames of Figure 2. It has been previously noted that, in inertial space, the trajectory of the CF frame approximates the CE frame to within 2% [*Dong et al.*, 1997]. Here I have shown that center of figure frames defined by no-net vector translation using only lateral displacements CL or only height displacements CH have trajectories that differ from that of CF by 13–27%.

[52] Geocenter differences (between frame types) have been noted experimentally. *Argus et al.* [1999] found secular geocenter variation at the 2–4 mm/yr level in differences of solutions in the CH, CL, and CM frames derived by *Watkins et al.* [1994], *Heki* [1996], *Argus* [1996], and C. Ma et al. (various GSFC VLBI solutions, 1993–1996). While some of these differences might be explained in terms of systematic error, to date there has been little theoretical explanation as to how real differences can arise from dynamic processes. The theory developed here at least provides a predictive basis for computing station coordinate variations that result from elastic crustal loading, using an approach that ought to be more broadly applicable. Even VLBI observations of quasars are indirectly sensitive to the geocenter associated with elastic loading, as the degree-one deformation that must accompany geocenter displacement changes the distances between radio telescopes [*Lavallée and Blewitt*, 2002].

[53] For completeness, note that the deformation relative to the inertial origin CM can always be constructed as the superposition of the deformation expressed in any frame B plus that frame's geocenter motion. Applying equation (15),

$$\begin{aligned} [\Delta s_h]_{CM} &= [\Delta s_h]_B + \hat{\mathbf{h}}.[\mathbf{t}_B]_{CM} \\ [\Delta s_l]_{CM} &= [\Delta s_l]_B + \hat{\mathbf{l}}.\,[\mathbf{t}_B]_{CM} \\ [\mathbf{t}_B]_{CM}[U]_{CM} &= [U]_B + g\hat{\mathbf{h}}.[\mathbf{t}_B]_{CM}. \end{aligned} \tag{27}$$

5. Conclusions

[54] Good geodetic practice and subsequent scientific interpretation requires self-consistency in dynamic models and terrestrial reference frame theory. This especially applies when attempting to separate global-scale, low-frequency processes and processes that involve degree-one deformation associated with surface load redistribution. Here I have identified the class of "isomorphic frames," which have the property that, under redistribution of the load characterized by load moment **m**, the origins translate relative to each other along the axis of **m**, and the deformation dynamics can be formulated using frame-dependent degree-one load Love numbers. The following frames have been shown to be isomorphic: center of mass of the solid Earth, center of mass of the entire Earth system, no-net translation of the surface, no-net horizontal translation of the surface, and no-net vertical translation of the surface. Degree-one load Love numbers in the CE frame commonly used for dynamic modeling are strictly not applicable to frames used in geodetic practice. The degree-one load Love number transformations and geocenter translations can be characterized by a single isomorphic parameter α, whose value can be derived, given the definition of a conceptual reference frame.

[55] As far as degree-one deformations are concerned, there is no fundamental distinction between the concepts of vertical motion versus horizontal motion, as they are shown to depend entirely on the choice of frame. Therefore, great care must be exercised when interpreting global-scale redistribution of surface loads, especially when the research question involves the concepts of vertical and horizontal. Examples demonstrate that peak surface displacements can occur where the magnitude of the load distribution is minimum, illustrating that local models of loading in the IERS conventions [*McCarthy*, 1996] are misleading for the investigation of global-scale phenomena.

[56] The degree-one displacement field does not look like a rigid body translation. Previous methods for determining the geocenter trajectory might be biased unless they also account for the sizable distance-changing deformation associated with change in the load center of mass. Errors in the rigid body translation model amount to ~30% at specific sites and up to ~40% in relative vectors. When using a geometrically-defined origin (CH) that minimizes height displacements, lateral deformations can be magnified by a factor of ~3 over those predicted in the CE frame, typically used by modelers, which is even larger than the height deformations in the CE frame. This type of effect might occur if a frame is realized by constraining the height of some stations around the globe. I therefore recommend that fiducial-free analysis be applied in geodesy, using this to construct coordinate time series in an appropriate realization of an isomorphic frame, with subsequent interpretation (or model inversion) using degree-one load Love numbers appropriate to the selected frame.

[57] Geocenter differences (between frame types) have been investigated experimentally, but predictive theory has been lacking. The theoretical development presented here quantitatively predicts differences in geocenter trajectories in different frames if there is a global-scale redistribution of surface mass. Differences in geocenter estimates, however, will generally also depend on solid Earth processes other than elastic loading, such as postglacial isostatic rebound, and they will depend on how well ideal reference frames have been realized by (imperfect) distributions of geodetic stations. Further development is needed along the lines of this research to better define a frame for studies of global climate change, for example, a frame to improve interpretation of global sea level rise. In particular, more theoretical development with the requisite clarity of concepts should in future clarify how, in principle, one can separate secular tectonics signals from decadal-scale loading, using decadal-scale spans of space geodetic data.

Appendix A

[58] The following identity was used to derive equation (13). The proof is simplified by choosing a spherical coordinate system (latitude φ, longitude λ) such that the polar axis points in the direction of **m**. By simple geometry and the definition of the surface gradient $\nabla = \hat{\varphi}\partial_\phi + \hat{\lambda}(1 \cos\phi)\partial_\lambda$, we can deduce

$$\begin{aligned} \hat{\mathbf{l}}.\nabla\left(\hat{\mathbf{h}}.\mathbf{m}\right) &= \hat{\mathbf{l}}.\left(\hat{\varphi}\partial_\phi + \hat{\lambda}(1/\cos\phi)\partial_\lambda\right)(m\sin\phi) \\ &= \hat{\mathbf{l}}.\hat{\varphi}m\cos\phi + 0 = \left(\hat{\mathbf{l}}.\hat{\varphi}\right)(m.\hat{\varphi}). \end{aligned} \quad \text{(A1)}$$

But we can also write the following, remembering that our choice of polar axis requires that $m_\lambda = 0$ everywhere:

$$\hat{\mathbf{l}}.\mathbf{m} = \hat{\mathbf{l}}.\left(\hat{\varphi}m_\phi + \hat{\lambda}m_\lambda + \hat{\mathbf{h}}m_h\right) = \left(\hat{\mathbf{l}}.\hat{\varphi}\right)(\mathbf{m}.\hat{\varphi}) + 0 + 0, \quad \text{(A2)}$$

hence proving the equivalence.

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G. Blewitt, University of Nevada, Nevada Bureau of Mines and Geology, Mail Stop 178, Reno, NV 89557, USA. (gblewitt@unr.edu)