Robustness analysis of 3D networks

Mustafa Berber*, Petr Vaníček, Peter Dare

Department of Geodesy and Geomatics Engineering, University of New Brunswick, P.O. Box: 4400, Fredericton, NB, E3B 5A3, Canada

Abstract

Traditionally, 3D geodetic networks are established as unions of horizontal and vertical control networks. However, over the last 10 years or so, GPS (Global Positioning System) networks have gained more and more importance. After geodetic networks are monumented, relevant measurements are taken, and point coordinates for the control points are estimated by the method of least squares. However, the method of least squares does not give any information about the robustness of networks. To measure robustness of a network, the deformation of individual points of the network is portrayed by strain. This technique is independent of adjustment constraints and reflects only the network geometry and accuracy of the observations. Furthermore, threshold values are needed to quantify the robustness of the network. If the displacements of some points of the network are worse than the threshold values, this suggests that we should redesign the network by changing the configuration or improving the measurements until we obtain a network of acceptable robustness. This paper describes how to obtain the displacements at the individual points of a 3D network, employs the specifications of Geodetic Survey Division to derive the acceptable values and shows the results on two different GPS networks.

1. Introduction

The first known application of strain analysis is due to Terada and Miyabe (1929) who used strain to describe deformation of the earth surface caused by earthquakes. Pope (1966), the first known geodesist to deal with strain analysis, also used this technique for application to repeated geodetic surveys to determine crustal movements. Vaníček et al. (1991) combined the reliability technique introduced by Baarda (1968) and the geometrical strength analysis method into one technique called robustness analysis. Vaníček et al. (2001) outlined robustness analysis of horizontal networks. Robustness analysis of geodetic networks is studied in Berber (2006).

In the past, geodetic networks used to be established separately as horizontal (2D) and vertical (height) networks (1D). However with the advance of technology, especially GPS (Global Positioning System), nowadays 3D geodetic networks are gaining more and more importance. To form 3D networks either horizontal information is combined with vertical information (to combine the 2D and 1D networks) or 3D terrestrial or satellite techniques such as GPS are used to enable 3D observations to be made.

The concept of strain (is defined as the ratio of increase in length to its original length) may be applied to the analysis of geodetic networks by considering the network to be a structure in itself. That is, stations are held together by the interconnecting observations like a building which is held together by its beams. In this analogy stations are considered to be the joints and observations are the beams and brackets. Distance observations can be thought of as beams of rigid length whose orientation in space is not fixed. Angles can be considered as brackets which fix the relative orientation (angles) between beams of different lengths. Azimuth can be thought of as a bracket that fixes the orientation of a beam with respect to the foundation, which acts as the “datum” (a datum is a defined surface which is used to determine the location of unknown points).

Using such an analogy helps to become familiar with the structure of geodetic networks. Furthermore, it helps to expand the robustness analysis (augmentation of traditional reliability analysis (Baarda’s approach see Baarda, 1968) with geometrical strength analysis is termed robustness analysis) technique to 3D. However, robustness analysis of 3D networks differs from the analysis of 2D networks. In such a manner strain is calculated as

\[ e = \frac{l - l'}{l} \]

where \( l \) is the deformed (in geodesy one really deals with a “potential deformation” that could be introduced by the undetected gross
errors or blunders in the observation, rather than deformation caused by, for example, physical movement of the points) length and \( l \) is the original length of the object. As mentioned above, in geodesy distances can be thought of as beams of rigid length. Hence, in 3D networks, one might run into a problem with heights. When two points have very nearly the same height (a common occurrence) the strain with respect to height might become extremely large and mislead the results since the height difference between the adjacent points will be very close to zero and this will make the denominator in Eq. (1) very close to zero. In other words, the configuration of a geodetic network is usually nearly two-dimensional, as the heights may not differ too much from each other. Then the strain in the dimension perpendicular to the surface may become ill-defined. This issue is addressed in Vanáček et al. (2008).

To overcome this problem finite strain could be used since the strain with respect to height might become very large. In Eq. (1), if the ratio of numerator to denominator becomes large, it is said that the strain is no longer infinitesimal strain; under these circumstances it becomes finite strain. The difference between finite strain and infinitesimal strain is that in finite strain the higher-order differentials of potential displacements are considered. However if any two points have exactly the same height, finite strain would not help because, no matter what the situation was, the result would be infinity. Moreover, it is investigated that the height from a surface such as ellipsoid or geoid could be defined but then the physical changes from that surface to the point of interest would not be explained. This is illustrated in Fig. 1. According to our consideration, in Fig. 1, the physical changes from the reference surface to point \( P_i \) need to be known. In this case, it is thought that instead of curvilinear coordinates, Cartesian coordinates (see Vanáček and Krakiwsky, 1986) can be used. The problem with the curvilinear coordinates is that when the points have the same height, the height difference between points relative to each other is zero. Since the magnitude (length) of the displacement vector is independent from the coordinate system, networks can be assessed in any coordinate system.

The maximum undetectable errors \( \Delta l \) (an error is the difference between a measured value and its true value) among the observations which would not be detected by a statistical test is given by Baarda (1968) as

\[
\Delta l = \sqrt{l_0 \frac{\sigma_i}{\sqrt{P_l}}}
\]

where \( \sqrt{l_0} \) is the value of the shift (non-centrality parameter) of the postulated distribution in the alternative hypothesis. \( \sqrt{l_0} \) is a function of both \( \alpha \) and \( \beta \). In this study 5% for \( \alpha \) and 5% for \( \beta \) are assumed (\( \alpha \) is the probability of committing a Type-I error and \( \beta \) is the probability of committing a Type-II error). Calculation of \( \sqrt{l_0} \) is given in Vanáček et al. (2001). \( \sigma_i \) is the a priori value of standard deviation of the \( i \)th observation and \( r_i \) is the redundancy number of the \( i \)th observation. Redundancy number is a number between 0 and 1 and it gives the “controllability” of the observation. If it is 1, it is said that the observation is very well controlled. If it is 0, it means that only the minimum number of observations are connecting the point to the rest of the network. This formula describes the existence of maximum undetectable errors due to the postulated distribution in the alternative hypothesis.

The least squares estimate for the displacements \( \Delta x \) caused by the maximum undetectable errors \( \Delta l \) in the observations is given by

\[
\Delta x = (A^T PA)^{-1} A^T P \Delta l
\]

where \( A \) is the design matrix and \( P \) is the weight matrix (Vanáček and Krakiwsky, 1986).

Nevertheless, the problem with the displacements is that their estimates are datum dependent. Therefore, the strain technique is used as it is independent of adjustment constraints and reflects only the network geometry and accuracy of the observations. Robustness analysis of 2D networks is studied in Berber et al. (2006). In this paper the robustness analysis technique has been expanded to be applicable to 3D networks; this approach is outlined in the following section.

2. Robustness analysis of 3D networks

Let us denote the displacement of a point \( P_i \) by

\[
\Delta x_i = \begin{bmatrix} \Delta x_i \\ \Delta y_i \\ \Delta z_i \end{bmatrix} = \begin{bmatrix} u_i \\ v_i \\ w_i \end{bmatrix}
\]

Then the tensor gradient with respect to position is (Love, 1944; Sokolnikoff, 1956; Timoshenko and Goodier, 1970):

\[
E_i = \begin{bmatrix} \frac{\partial u_i}{\partial x} & \frac{\partial u_i}{\partial y} & \frac{\partial u_i}{\partial z} \\ \frac{\partial v_i}{\partial x} & \frac{\partial v_i}{\partial y} & \frac{\partial v_i}{\partial z} \\ \frac{\partial w_i}{\partial x} & \frac{\partial w_i}{\partial y} & \frac{\partial w_i}{\partial z} \end{bmatrix}
\]

For \( \forall j = 0, 1, \ldots, t \) (\( t \) is the number of connections) the displacements \( u, v \) and \( w \) can be calculated as follows:

\[
a_i + \frac{\partial u_i}{\partial x}(X_i - X_j) + \frac{\partial u_i}{\partial y}(Y_i - Y_j) + \frac{\partial u_i}{\partial z}(Z_i - Z_j) = u_j
\]

\[
b_i + \frac{\partial v_i}{\partial x}(X_i - X_j) + \frac{\partial v_i}{\partial y}(Y_i - Y_j) + \frac{\partial v_i}{\partial z}(Z_i - Z_j) = v_j
\]

\[
c_i + \frac{\partial w_i}{\partial x}(X_i - X_j) + \frac{\partial w_i}{\partial y}(Y_i - Y_j) + \frac{\partial w_i}{\partial z}(Z_i - Z_j) = w_j
\]

where all the partial derivatives as well as the absolute terms \( a_i, b_i, c_i \) and the coordinates \( X_i, Y_i \) and \( Z_i \) refer to point \( P_i \) and point \( P_j \) is connected (by an observation) to the point of interest, point \( P_i \). In matrix form:
∀i in the network \[ K_i \begin{bmatrix} \frac{\partial u_i}{\partial x} \\ \frac{\partial v_i}{\partial y} \\ \frac{\partial w_i}{\partial z} \end{bmatrix} = \begin{bmatrix} a_i \\ b_i \\ c_i \end{bmatrix}, \quad K_i = \begin{bmatrix} \frac{\partial v_i}{\partial x} \\ \frac{\partial w_i}{\partial y} \\ \frac{\partial x_i}{\partial z} \end{bmatrix} \]

∀i \in \{1, \ldots, n\}; n is the number of points in the network. If these three equations are solved using the method of least squares, we obtain

\[
\begin{align*}
\begin{bmatrix} a_i \\ b_i \\ c_i \end{bmatrix} &= (K_i^T K_i)^{-1} K_i^T \begin{bmatrix} u_i \\ v_i \\ w_i \end{bmatrix}, \\
\begin{bmatrix} a_i \\ b_i \\ c_i \end{bmatrix} &= \begin{bmatrix} \frac{\partial v_i}{\partial x} \\ \frac{\partial w_i}{\partial y} \\ \frac{\partial x_i}{\partial z} \end{bmatrix}.
\end{align*}
\]

Assembling them into a hypermatrix and using Eq. (5), we get

∀i in the network \[ \text{vec}(E_i) = T_i \Delta x_i, \]

Substituting Eq. (3) in Eq. (10), we write

∀i in the network \[ \text{vec}(E_i) = T_i (A^T P A)^{-1} A^T P \Delta \]

To determine the displacements of point Pi, we introduce the “initial conditions” \(X_0, Y_0\), and \(Z_0\). Initial conditions are the coordinates which are obtained minimizing the norm of the displacement vectors at all points in the network. This means that to calculate \(X_0, Y_0\), and \(Z_0\), the displacements in the network points should be minimized (see Appendix A). Once \(X_0, Y_0, Z_0\) have been determined \(u_i, v_i, w_i\) are calculated from:

\[
\begin{bmatrix} u_i \\ v_i \\ w_i \end{bmatrix} = (K_i^T K_i)^{-1} K_i^T \begin{bmatrix} X_i - X_0 \\ Y_i - Y_0 \\ Z_i - Z_0 \end{bmatrix}
\]

After computing the displacements \(u, v\) and \(w\) for each point in the network, we can calculate the amount of total displacement at each point from:

\[
d_i = \sqrt{u_i^2 + v_i^2 + w_i^2}
\]

Fig. 2 outlines the procedure for robustness analysis. In geology or geophysics one may be interested in measuring the deformations using strain on a local scale such as engineering applications or regional and global scale such as crustal dynamics analysis. Given the role that geodetic measurements (GPS time series) now play in geophysics and geodynamics, it seems that the outlined approach might be one of broad interest and considerable importance for the geodynamics community.

3. Determination of threshold values

In this paper we propose to use the Accuracy Standards for Positioning given by Geodetic Survey Division (GSD), (1996) to compute threshold values. These threshold values are going to enable us to “measure” the robustness of any network.

After least squares method is performed, confidence ellipses are derived from the covariance matrix. They provide a graphical means of viewing the results of a network adjustment. The standard confidence ellipse representing the one-sigma network accuracy of the adjusted horizontal coordinates at point Pi, is defined by its major (a) and minor (b) semi-axes. Using the elements of the covariance matrix, semi-major and semi-minor axes of the standard confidence ellipse can be computed respectively Mikhail and Gracie (1981):

\[
\sigma_a = [(\sigma_x^2 + \sigma_y^2)/2 + q_i]^{1/2}
\]

\[
\sigma_b = [(\sigma_x^2 + \sigma_y^2)/2 - q_i]^{1/2}
\]

where

\[
q_i = [(\sigma_x^2 - \sigma_y^2)/2 + \sigma_z^2]^{1/2}
\]

and where \(\sigma_x^2\) is the variance of the X coordinate (m²); \(\sigma_y^2\) is the variance of the Y coordinate (m²); \(\sigma_z^2\) is the covariance between X and Y coordinates (m²²).

Since a displacement is normally regarded as a vector, in this study two vectors (the vector which is computed using Eq. (13) and the vector which is going to be calculated using Eq. (19)) are compared.

To obtain semi-axes of the 95% confidence ellipse, we write

\[
\sigma_{95a} = 2.45 \sigma_a
\]

\[
\sigma_{95b} = 2.45 \sigma_b
\]

The 95% confidence interval representing the network accuracy of the ellipsoidal height is obtained by multiplying \(\sigma_h\) (standard deviation of the ellipsoidal height in units of metres) which is extracted from the covariance matrix by the expansion factor 1.96 for a single variate probability distribution.

Therefore in this study the following formula is implemented:

\[
d_i = \sqrt{u_i^2 + v_i^2 + w_i^2}
\]

where \(\sigma_{95a}\) is the semi-major axis of the 95% confidence ellipse, \(\sigma_{95b}\) the semi-minor axis of the 95% confidence ellipse, \(\sigma_{95h}\) the 95% confidence interval of height component and \(\delta_1\) is the threshold value which the displacements are compared against. In this equation the horizontal semi-axes must be scaled by \((2.795/2.447)\) and the vertical interval by \((2.795/1.960)\). So first one needs to replace the 2D expansion factor and the 1D expansion factor with the 3D expansion factor before forming the 3D limit which is an approximation of the 3D confidence ellipsoid.
The confidence ellipsoid could be used to determine the accuracy of the adjusted coordinates of network points for GPS networks since one may have the full covariance matrix for GPS observations. However, with the traditional approach, horizontal and vertical coordinates are obtained separately, so generally a full covariance matrix is not available for the points in the classical (terrestrial) three-dimensional networks. Therefore, in this study, the general case which has suggested by GSD of Canada is implemented.

Here the displacements are calculated using Eq. (13) to be able to compare them with the threshold values. The threshold values for each point in the network are computed using Eq. (19). Since the magnitude (length) of the displacement vector is independent of the coordinate system, networks can be assessed in any coordinate system.

4. Examples

Two GPS networks are examined here. The first network is called simple GPS network (a real network) and is shown in Fig. 3. It is a small network which consists of seven points, one of which (point 1) is fixed, and 42 coordinate differences. The range of the baseline component standard deviations are 0.7–3.1 mm.
As can be seen from Fig. 3, some of the displacements are bigger at the edge of the network. However, if the redundancy number of the observations increases, the displacements get smaller. For example, at point 5 there are five connections whereas at point 1 there are three connections and consequently the displacement is smaller for point 5. Standard deviations of observations in this network are quite small therefore the displacements are quite small.

Displacements are computed using Eq. (13) and plotted in Fig. 3. The threshold values are calculated using Eq. (19). Then the displacements are compared with the threshold values, these comparisons are given in Table 1. Since for ∀ in the network displacement values are less than the threshold values, this is a robust network.

The second network is called Northwest Territories Network; it is shown in Fig. 4. This is a real GPS network too. It consists of 33 points, one of which (point 1) has been held fixed in the adjustment, and 402 coordinate differences. The range of the baseline component standard deviations is 8–774 mm; these values look quite large since it is a rather old GPS network.

Table 1
Displacements and threshold values for simple GPS network

<table>
<thead>
<tr>
<th>Points</th>
<th>$d_i$ (m)</th>
<th>$\delta_i$ (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.002</td>
<td>0.005</td>
</tr>
<tr>
<td>3</td>
<td>0.000</td>
<td>0.005</td>
</tr>
<tr>
<td>4</td>
<td>0.002</td>
<td>0.006</td>
</tr>
<tr>
<td>5</td>
<td>0.000</td>
<td>0.005</td>
</tr>
<tr>
<td>6</td>
<td>0.001</td>
<td>0.005</td>
</tr>
<tr>
<td>7</td>
<td>0.002</td>
<td>0.007</td>
</tr>
</tbody>
</table>

As can be seen from Fig. 4, generally the displacements are bigger at the edge of the network since the control of these points is more limited compared to the other points in the network. However, as soon as the control increases the displacements get smaller. For example, at point 9 there are three connections whereas at point 20 there are four connections; consequently, the displacement is smaller at point 20.

It is seen from detailed analysis of the original observations that points 5 and 22 have some observations which have large standard deviations. These less precise observations affect the points in their vicinity; therefore we get a large displacement at point 22.

Table 2
Displacements and threshold values for Northwest Territories network

<table>
<thead>
<tr>
<th>Points</th>
<th>$d_i$ (m)</th>
<th>$\delta_i$ (m)</th>
<th>Points</th>
<th>$d_i$ (m)</th>
<th>$\delta_i$ (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.91</td>
<td>0.14</td>
<td>*18</td>
<td>0.10</td>
<td>0.13</td>
</tr>
<tr>
<td>3</td>
<td>0.07</td>
<td>0.14</td>
<td>*19</td>
<td>0.25</td>
<td>0.19</td>
</tr>
<tr>
<td>*4</td>
<td>0.23</td>
<td>0.14</td>
<td>*20</td>
<td>0.28</td>
<td>0.15</td>
</tr>
<tr>
<td>5</td>
<td>0.09</td>
<td>0.12</td>
<td>*21</td>
<td>0.20</td>
<td>0.13</td>
</tr>
<tr>
<td>6</td>
<td>0.05</td>
<td>0.15</td>
<td>*22</td>
<td>0.63</td>
<td>0.12</td>
</tr>
<tr>
<td>*7</td>
<td>0.17</td>
<td>0.15</td>
<td>23</td>
<td>0.13</td>
<td>0.23</td>
</tr>
<tr>
<td>8</td>
<td>0.14</td>
<td>0.15</td>
<td>*24</td>
<td>0.28</td>
<td>0.23</td>
</tr>
<tr>
<td>*9</td>
<td>0.93</td>
<td>0.25</td>
<td>*25</td>
<td>0.18</td>
<td>0.15</td>
</tr>
<tr>
<td>10</td>
<td>0.14</td>
<td>0.14</td>
<td>26</td>
<td>0.07</td>
<td>0.08</td>
</tr>
<tr>
<td>11</td>
<td>0.14</td>
<td>0.25</td>
<td>27</td>
<td>0.08</td>
<td>0.14</td>
</tr>
<tr>
<td>*12</td>
<td>0.25</td>
<td>0.17</td>
<td>28</td>
<td>0.11</td>
<td>0.13</td>
</tr>
<tr>
<td>13</td>
<td>0.12</td>
<td>0.12</td>
<td>*29</td>
<td>1.02</td>
<td>0.27</td>
</tr>
<tr>
<td>14</td>
<td>0.02</td>
<td>0.10</td>
<td>*30</td>
<td>0.76</td>
<td>0.13</td>
</tr>
<tr>
<td>*15</td>
<td>0.12</td>
<td>0.11</td>
<td>*31</td>
<td>0.54</td>
<td>0.12</td>
</tr>
<tr>
<td>16</td>
<td>0.03</td>
<td>0.08</td>
<td>32</td>
<td>0.13</td>
<td>0.15</td>
</tr>
<tr>
<td>17</td>
<td>0.01</td>
<td>0.06</td>
<td>33</td>
<td>0.09</td>
<td>0.13</td>
</tr>
</tbody>
</table>
Also, points 12, 21 and 24 have some observations which have large standard deviations. Consequently, these observations affect that area of the network and we get some significant displacements at these points. At point 29 not only is the control lower but also the observations have large standard deviations; therefore, we obtain the largest displacement at this point. Most of the other points are in the middle of the network therefore their control is quite high compared to the other points at the edge of the network so we do not get large displacements at these points.

Displacements are computed using Eq. (13) and plotted in Fig. 4. The threshold values are calculated using Eq. (19). The comparison of the displacement values with the threshold values is given in Table 2. In this network for some points displacements are larger than the threshold values so the network is not all robust at the required level and it needs to be improved. These weak points are identified in Table 2 by an asterisk.

5. Conclusions

In order to be able to calculate displacements in 3D networks, initial conditions must be computed. Furthermore, threshold values are needed to evaluate 3D networks. These threshold values are going to enable us to assess the robustness of 3D networks. In this study, “Accuracy Standards for Positioning” given by Geodetic Survey Division are used to compute threshold values for 3D networks. The numerical results prove that this approach works well.

If a network has some defects, the robustness analysis technique reveals them and portrays them. When the controllability of the network points is lower we obtain bigger displacements. However, as soon as the controllability increases the displacements get smaller. If some observations have large standard deviations they affect the points in the vicinity and therefore we get big displacements at these points. If a network has some deficiency (such as the standard deviations are rather large) we can determine the weakness of the network for these points. If a network is not robust we must redesign the network by changing the configuration or improving the measurements until we obtain a network of acceptable robustness.

The advantages of robustness analysis technique are that due to its shortcomings Baarda’s method might not detect all the possible errors. In addition, when the results of the Baarda’s method are used to calculate the displacements, the outcomes are not datum independent. However, robustness of a network should depend only on the network geometry and accuracy of the observations. Therefore, the strain technique is employed. Thus, robustness analysis reveals and portrays the defects of networks datum independently.

Acknowledgements

We thank the Canadian “Natural Sciences and Engineering Research Council” (NSERC) for funding this work.

We thank Dr. Mike Craymer in Geodetic Survey Division of Canada for providing the network examples.

Appendix A

Eq. (12) is a system of first order differential equations. In order to solve the system, it should be integrated. Therefore the initial conditions \((X_0, Y_0, Z_0)\) have to be determined. In order to be able to calculate the initial conditions, the displacements caused by maximum undetectable errors in network points should be minimized. This means that the norm of the displacement vectors for all points in the network should be minimum, i.e.,

\[
\min_{(X_0, Y_0, Z_0 \in \mathbb{R})} \sum_{i=1}^{n} ||\Delta r_i|| = \min_{(X_0, Y_0, Z_0 \in \mathbb{R})} \sum_{i=1}^{n} (u_i^2 + v_i^2 + w_i^2) \tag{A1}
\]

Here we are looking for the relation between the initial conditions and the strain parameters, therefore the absolute terms \(a_i, b_i\) and \(c_i\) are no interest to us. So, if the absolute terms are removed from
Eq. (6), and then if these reduced equations are employed, we get
\[ d_1 = \sum_{i=1}^{n} \left[ \left( \frac{\partial u_i}{\partial x} \right) X_i + \left( \frac{\partial v_i}{\partial y} \right) + \frac{\partial w_i}{\partial z} \right] \]

\[ + \left( \frac{\partial u_i}{\partial y} \right) Y_i \]

\[ + \left( \frac{\partial u_i}{\partial z} \right) Z_i \]  \hspace{1cm} (A10)

If the same equation is differentiated with respect to \( Y_0 \), we can write
\[ d \sum_{i=1}^{n} ||\Delta \vec{r}_i||_{l_2} \frac{\partial Y_0}{\partial Y_0} = 0 \]  \hspace{1cm} (A11)

We get
\[ \sum_{i=1}^{n} \left[ \left( \frac{\partial u_i}{\partial x} \right) - \frac{\partial u_i}{\partial y} \right] X_i \]

\[ + \left( \frac{\partial v_i}{\partial x} \right) - \frac{\partial v_i}{\partial y} \right] Y_i \]

\[ + \left( \frac{\partial w_i}{\partial x} \right) - \frac{\partial w_i}{\partial y} \right] Z_i \]  \hspace{1cm} (A5)

To simplify this equation it can be expressed in the following form,
\[ \sum_{i=1}^{n} \left[ \frac{\partial u_i}{\partial x} \right] X_i \right] \]  \hspace{1cm} (A6)

We can write
\[ a_1 = \sum_{i=1}^{n} \left[ \left( \frac{\partial u_i}{\partial x} \right)^2 + \left( \frac{\partial v_i}{\partial x} \right)^2 + \left( \frac{\partial w_i}{\partial x} \right)^2 \right] \]  \hspace{1cm} (A7)

\[ b_1 = \sum_{i=1}^{n} \left[ \frac{\partial u_i}{\partial y} + \frac{\partial v_i}{\partial y} + \frac{\partial w_i}{\partial y} \right] \]  \hspace{1cm} (A8)

\[ c_1 = \sum_{i=1}^{n} \left[ \frac{\partial u_i}{\partial z} + \frac{\partial v_i}{\partial z} + \frac{\partial w_i}{\partial z} \right] \]  \hspace{1cm} (A9)
write
\[ \sum_{i=1}^{n} \left[ \left( \frac{\partial u_i}{\partial x} - \frac{\partial v_i}{\partial y} \right) \mathbf{A}_i \right] = 0 \]

We get
\[
\sum_{i=1}^{n} \left[ \left( \frac{\partial u_i}{\partial x} \frac{\partial v_i}{\partial y} \frac{\partial w_i}{\partial z} \right) \mathbf{A}_i \right] = 0
\]

To simplify this equation it can be expressed in the following form,
\[
\sum_{i=1}^{n} \left( a_i \mathbf{X}_0 + b_i \mathbf{Y}_0 + c_i \mathbf{Z}_0 + d_i \mathbf{Z}_0 \right) = 0
\]

We can write
\[
d_3 = \sum_{i=1}^{n} \left( \frac{\partial u_i}{\partial x} \frac{\partial v_i}{\partial y} \frac{\partial w_i}{\partial z} \right) = 0
\]

If these Eqs. (A6), (A13) and (A20) are solved with the compact form, we obtain the initial conditions \( \mathbf{X}_0, \mathbf{Y}_0, \mathbf{Z}_0 \) as follows
\[
\begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \end{bmatrix} = \mathbf{A}^{-1} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}
\]

The initial conditions \( \mathbf{X}_0, \mathbf{Y}_0, \mathbf{Z}_0 \) are substituted in Eq. (12) to calculate the displacements \( u, v, w \) for each point in the network.

References


