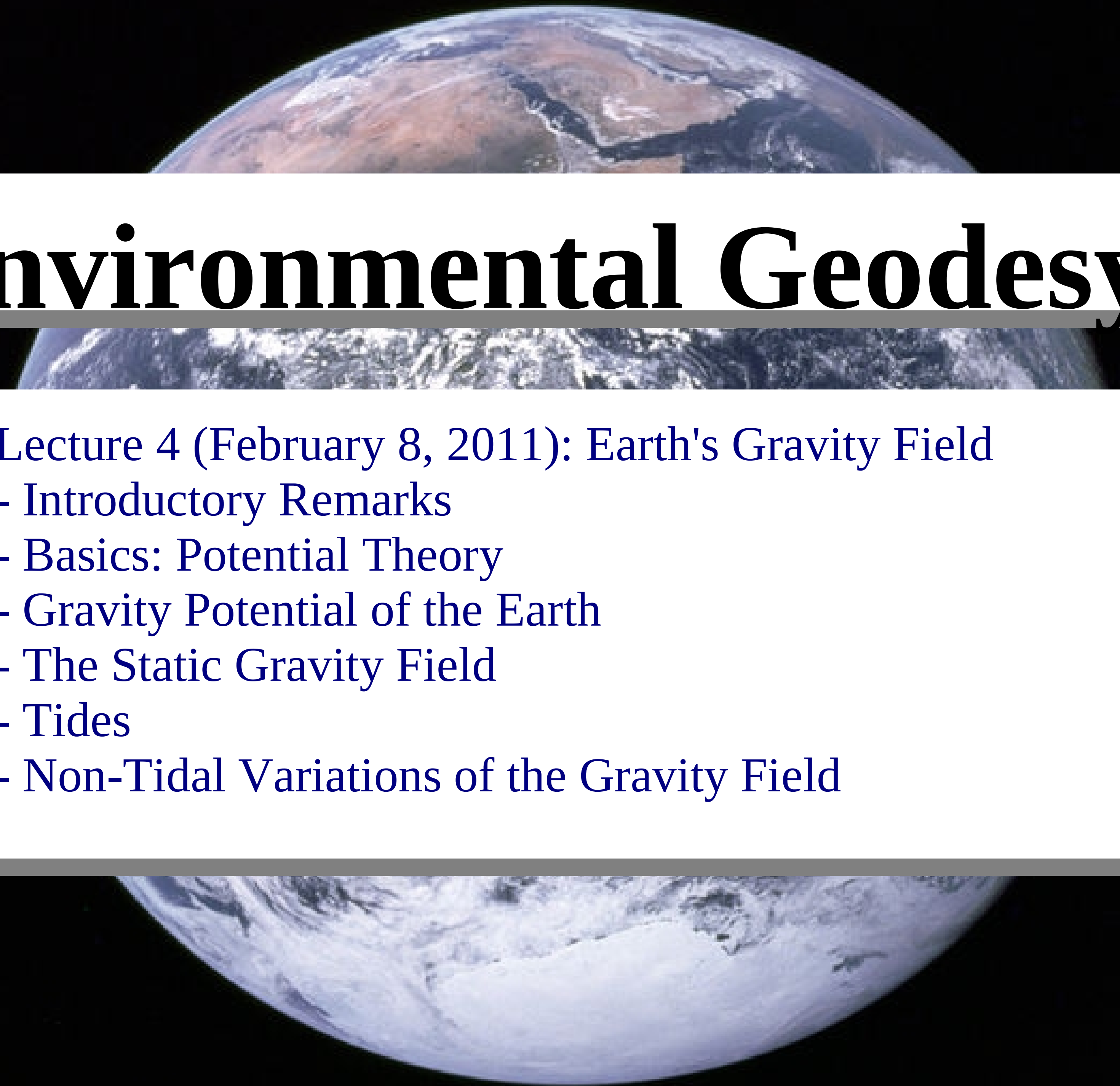


Environmental Geodesy

Lecture 4 (February 8, 2011): Earth's Gravity Field
- Introductory Remarks
- Basics: Potential Theory
- Gravity Potential of the Earth
- The Static Gravity Field
- Tides
- Non-Tidal Variations of the Gravity Field

Introductory Remarks

Why is the gravity field of importance?

- "elevation above sea level" is, in fact, "elevation above geoid"
- "horizontal" and "vertical" relate to the gravity field
- many geodetic instruments use gravity as reference
- gravity measurements contain information on the Earth's interior:
  - the static (time-independent) gravity field relates to the strength within the Earth, the composition near the Earth's surface, and slow dynamical processes within the Earth.
  - the time-variable gravity field relates to mass transport and tides

Introductory Remarks

Some application of the static gravity field:

- mass of the Earth;
- height systems (orthometric heights, linking national and regional height systems);
- exploration (minerals, oil, groundwater, ...);
- geodynamics (convection and plate tectonics, isostasy, postglacial rebound);
- ocean circulation (dynamic sea surface topography);

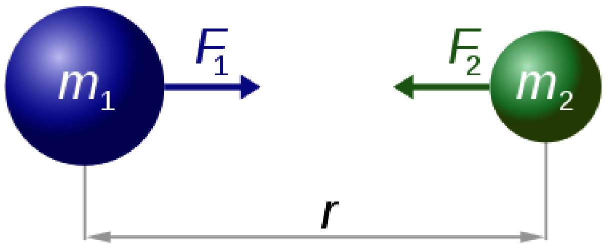
Some application of the time-variable gravity field:

- solid Earth tides ocean tidal loading;
- free oscillations of the Earth;
- groundwater and soil moisture changes (in situ);
- mass transport in the fluid envelope of the Earth;

Basics: Potential Theory

Physical Theory:
Newton's law of gravitation:
Force F between two point masses, m_1 and m_2, separated by a distance r is:

$$F_1 = F_2 = \frac{G m_1 m_2}{r^2}$$



where G is the universal "gravitational constant".
$G = 6.67\times10^{-11}\ \mathrm{m^3\ kg^{-1}\ s^{-2}}$.

General Theory of Relativity:
- effects important at the 10^{-9} level

Basics: Potential Theory

G has been measured for more than 300 years;
Still large uncertainties.

Force F on a point masses m, separated by a distance r from the (spherical) Earth with mass M_E and is:

$$F(r) = \frac{GM_{\mathrm{E}}m_2}{r^2}$$

Measurement of GM_E is still an issue (will be addressed later)

GM_E determines the period T of (small) satellites:

$$T = 2\pi\sqrt{\frac{r^3}{GM_{\mathrm{E}}}}$$

Basics: Potential Theory

Gravitational force exerted on a small mass *m* at position ***x*** outside of Earth by a small mass element *dm'* at position ***x'*** in Earth:

$$df(x) = \frac{Gmdm'}{||x - x'||^2} = \frac{Gmdm'}{b^2}$$

Infinitesimal gravitational *dg* acceleration due to *dm*:

$$dg(x) = \frac{df(x)}{m} = \frac{Gdm'}{b^2}$$

Gravitational acceleration *g*:

$$g(x) = \int_{\text{mass}} \frac{G\rho(x')(x' - x)}{||x - x'||^3} d^3x'$$

Gravitational potential ϕ:

$$\nabla\Phi(x) = g(x)$$

$$\Phi(x) = \int_{\text{mass}} \frac{G\rho(x')}{||x - x'||} d^3x'$$

- Integral form of Newton's Law of Gravity

Basics: Potential Theory

Nabla operator: $$\nabla\Phi(\vec{x}) = \partial_x\Phi\vec{e}_x + \partial_y\Phi\vec{e}_y + \partial_z\Phi\vec{e}_z$$

Gravitational potential ϕ: $$\nabla\Phi(x) = g(x)$$

$$\Phi(x) = \int_{\text{mass}} \frac{G\rho(x')}{||x - x'||} d^3x'$$

This is the integral form of Newton's Law of Gravity.

Basics: Potential Theory

Laplace operator: $$\nabla^2\Phi(\vec{x}) = \partial_x^2\Phi + \partial_y^2\Phi + \partial_z^2\Phi$$

Poisson's equation: $$\nabla^2\Phi(\vec{x}) = -4\pi G\rho(\vec{x})$$

- Newton's Law of Gravity implies Poisson's equation
- Poission's equation implies Newton's Law of Gravity

Proof involves use of Green's Functions
- Green's Function of Poisson's equation:

$$\frac{GM}{||\vec{x}-\vec{x}'||}$$

Continuity conditions for density discontinuities:

$$\Phi_1 = \Phi_2$$

$$\hat{n}\cdot\nabla\Phi_2 - \hat{n}\cdot\nabla\Phi = -4\pi G\delta\rho$$

Basics: Potential Theory

Gauss' Law: $\int_S \hat{n} \cdot \vec{g} dS = -4\pi G \int_V \rho dV$

The integral over the surface of the normal component of *g* is proportional to the total mass inside the surface.

Region with no mass density:
Laplace equation:

$$\nabla^2 \Phi(\vec{x}) = 0$$

In most cases, a combination of Newton's Law of Gravity and the Lapace equation is sufficient to solve the problem.

Gravity Potential of the Earth

Gravitational potential for a uniform sphere:

$$\Phi(x) = \frac{GM}{d}$$

Good approximation for the Earth: g ~ 9.82 ms^{-2}

Gravity potential of a rotating elliptical body:
For a orthogonal coordinate system with the Z-axis coinciding with the rotation axis of the Earth, and origin in the Center of Mass (CM):
Integral form:

$$\Phi(\vec{x}) = \int_{\text{mass}} \frac{G\rho(\vec{x}')}{||\vec{x} - \vec{x}'||} d^3x' + \frac{1}{2}(x^2 + y^2)\omega^2$$

Poisson's equation:

$$\nabla^2\Phi(\vec{x}) = -4\pi G\rho(\vec{x}) + 2\omega^2$$

Laplace equation:

$$\nabla^2\Phi(\vec{x}) = 2\omega^2$$

Gravity Potential of the Earth

Solving the Laplace equation if the potential or the acceleration normal to the boundary is given on the boundary:
- find a set of solutions that satisfy:

$$\nabla^2 \Phi(\vec{x}) = 0$$

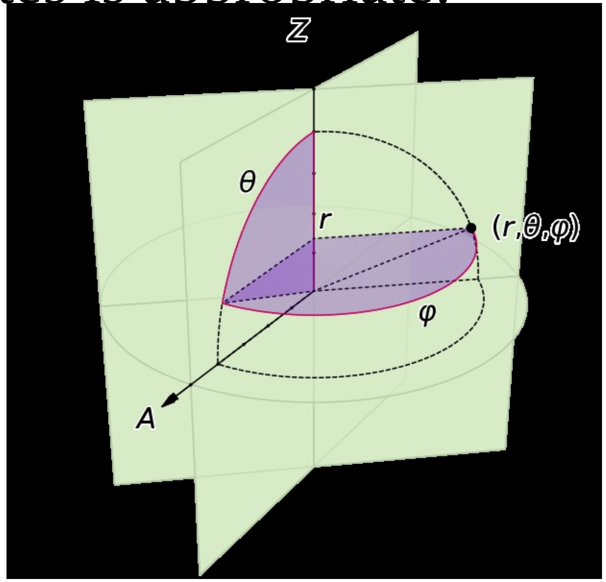
- combine the solutions to satisfy the boundary conditions
- combination still satisfy Laplace equation due to linearity of the Laplace operator.

For spherical boundaries, use of spherical coordinates is appropriate:

$$x = r \sin\theta \cos\varphi$$

$$y = r \sin\theta \sin\varphi$$

$$z = r \cos\theta$$



Gravity Potential of the Earth

For spherical boundaries, use of spherical coordinates is appropriate:
Laplace operator:

$$\nabla^2\Phi = \frac{1}{r}\partial_r(r^2\partial_r\Phi) + \frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\partial_\theta\Phi) + \frac{1}{r^2\sin^2\theta}\partial_\varphi^2\Phi$$

Looking for separable solutions:

$$\Phi(r,\theta,\varphi) = T(\theta)L(\varphi)R(r)$$

General solution has the form:

$$\Phi(r,\theta,\varphi) = \left\{r^l r^{-(l+1)}\right\} \times Y_l^m(\theta,\varphi)$$

$$Y_l^m(\theta,\varphi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)e^{im\varphi}$$

$$Y_l^m \equiv (-1)^m Y_l^{(-m)*}$$

Gravity Potential of the Earth

For a given density distribution, find potential outside a sphere including all mass:

Expansion of density in spherical harmonics:

$$\rho(\vec{x'}) = \sum_{l',m'} \rho_{l'}^{m'}(r') Y_{l'}^{m'}(\theta', \varphi')$$

Potential:

$$\Phi(\vec{x}) = 4\pi G \sum_{l,m} \frac{Y_l^m(\theta,\varphi)}{2l+1} \frac{1}{r^{l+1}} \left[\int (r')^{l+2} \rho_l^m(r') dr' \right]$$

Gravity Potential of the Earth

For a more realistic rotating Earth with arbitrary aspherical shape the potential has the form:

$$\Phi_{\mathrm{T}} = \left[\frac{GM}{r} + \frac{1}{3}\Omega^2 r^2\right] - \left[\frac{1}{3}\Omega^2 r^2 + \frac{GM}{r^3} a^2 J_2\right] P_2(\cos\theta)$$
$$- \frac{GM}{r} \sum_{l=2}^{\infty} \sum_{m=-l;(l,m)\neq(2,0)}^{l} \Phi_l^m Y_l^m(\theta,\varphi) \left(\frac{a}{r}\right)^l$$

Geoid height above reference ellipsoid:

$$N = \left.\frac{\Delta\Phi}{g_{\mathrm{T}}}\right|_{r=a}$$

$$\Delta\Phi = -\frac{GM}{r} \sum_{l=2}^{\infty} \sum_{m=-l;(l,m)\neq(2,0)}^{l} \Phi_l^m Y_l^m(\theta,\varphi) \left(\frac{a}{r}\right)^l$$

Gravity Potential of the Earth

Standard formula for geoid:

$$\Phi = \frac{GM}{r}\left(1 + \sum_{n=2}^{n_{max}} \left(\frac{a}{r}\right)^n \sum_{m=0}^{n} P_{nm}(\sin\phi)\left[C_{nm}\cos m\lambda + S_{nm}\sin m\lambda\right]\right)$$

Determination of geoid:
Stokes formula:

$$N = -\frac{a}{2\pi g_{\mathrm{T}}|_a} \int \Delta g(\theta', \varphi') f(\gamma) \sin\theta' d\theta' d\varphi'$$

For satellite measurements:

$$N = -a \sum_{l,m} \Phi_l^m Y_l^m$$

Static Gravity Field

Standard formula for geoid:

$$\Phi = \frac{GM}{r}\left(1 + \sum_{n=2}^{n_{max}} \left(\frac{a}{r}\right)^n \sum_{m=0}^{n} P_{nm}(\sin\phi)\left[C_{nm}\cos m\lambda + S_{nm}\sin m\lambda\right]\right)$$

Effect	Source	Magnitude ($1\ \mathrm{mGal} = 10^{-5}\ \mathrm{m \cdot s^{-2}}$)
Main	Total mass of Earth, Oblate shape, Earth Rotation	980,000 - 990,000 mGal
Elevation	Distance from centre of Earth	0.3 mGal/m
Crustal	Density variations in rocks within the Earth's crust	$<$500 mGal [usually 10's mGal]
Tides	Sun and moon	$<$1 mGal

Tides

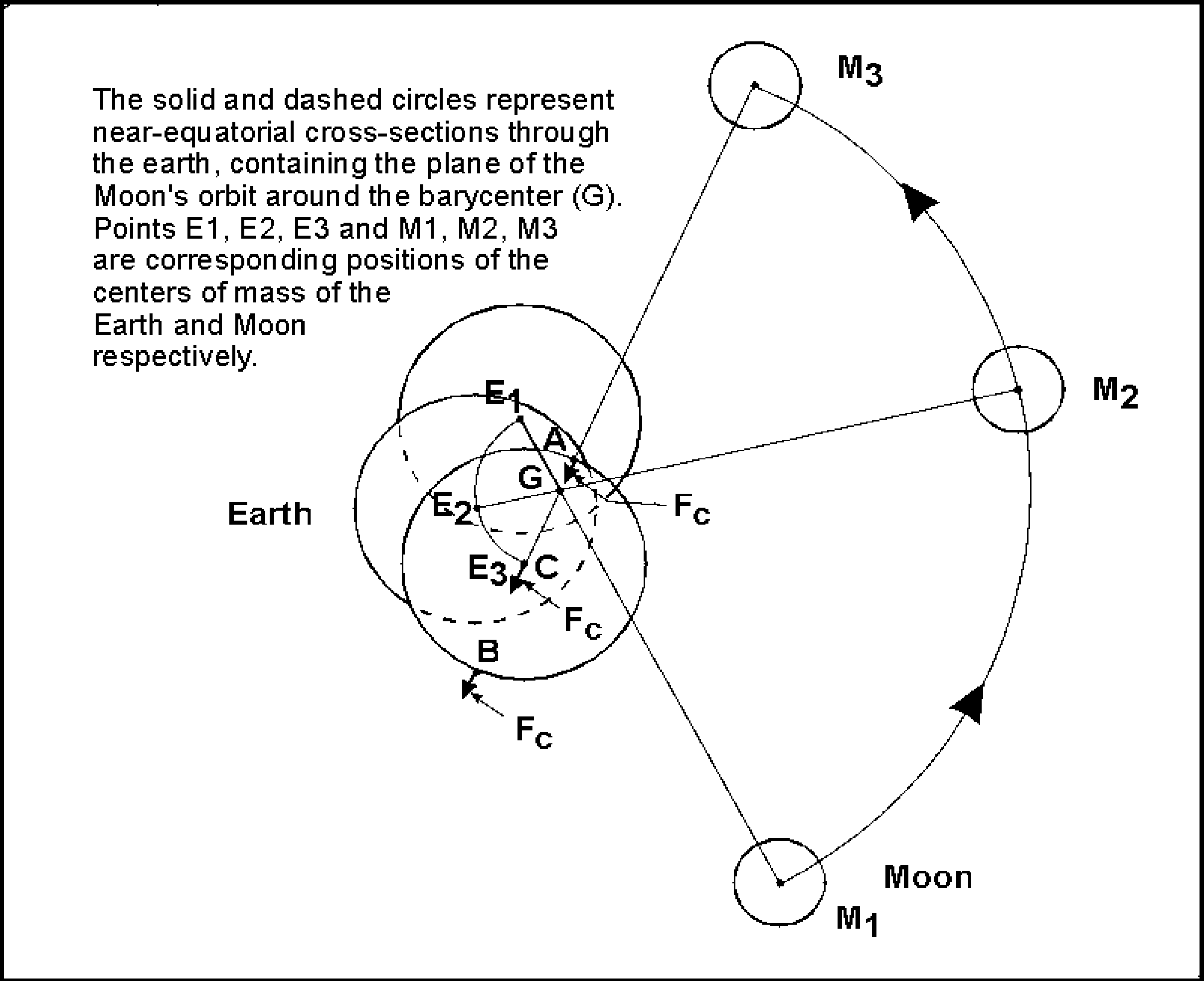
Why are tides of relevance to geodesy?

- Solid Earth tide change the shape, gravity field and rotation of the Earth;
- The response to tidal forces contains information on the viscoelastic properties of the Earth and the rotational dynamics;
- Tides contribute to station motion and need to be taken into account in geodetic analyses in the station motion model:

$$\vec{x}(t) = \vec{x}_0 + \sum_{k=1}^{K} \delta\vec{x}_k(t)$$

- Ocean tides change sea level and sea surface height;
- Ocean tidal loading contributes to station motion;
- Response to ocean tidal loading contains information on crustal structure.

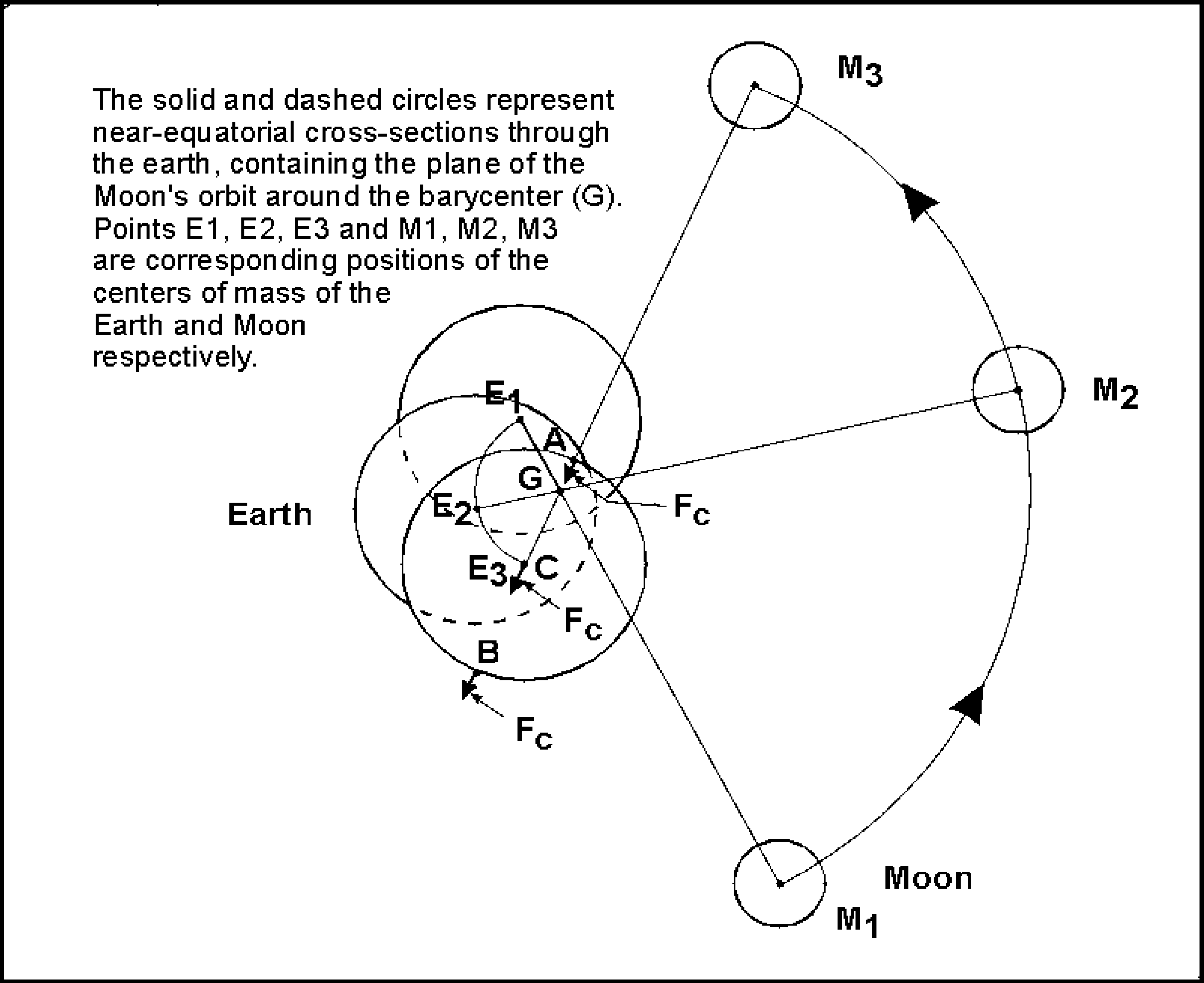
Tides



Origin of tidal forces in a two-body system: Revolution (without rotation) around the barycenter of the two bodies.

- Centrifugal force due to revolution is homogeneous throughout the Earth/Moon
- Attraction of the Moon is not homogeneous
- Difference between centrifugal and gravitational forces is the tidal force.

Tides



Origin of tidal forces in a two-body system: Revolution (without rotation) around the barycenter of the two bodies.

- Period of revolution is one months
- Period of rotation of the moon is one months: tidal bulge is looked
- Period of rotation of the Earth is one day; tidal bulge moves through the solid Earth.

Tides

Centrifugal force due to orbital motion around barycenter is compensated by gravitational force of the second body:

$$\vec{a}_0 = \frac{GM_b}{s^2}\frac{\vec{s}}{s}$$

$\vec{a}_0$: homogeneous centrifugal force in body 1 due to revolution around barycenter;
M_b: mass of the body 2;
$\vec{s}$: distance between the centers of the two bodies.

Acceleration at point P is the difference between homogeneous centrifugal forces and inhomogeneous gravitational force:

$$\vec{b} = \vec{a}_p - \vec{a}_0 = \frac{GM_b}{d^2} \cdot \frac{\vec{d}}{d} - \frac{GM_b}{s^2}\frac{\vec{s}}{s}$$

$\vec{d}$: distance between point P and the center of body 2.

Tides

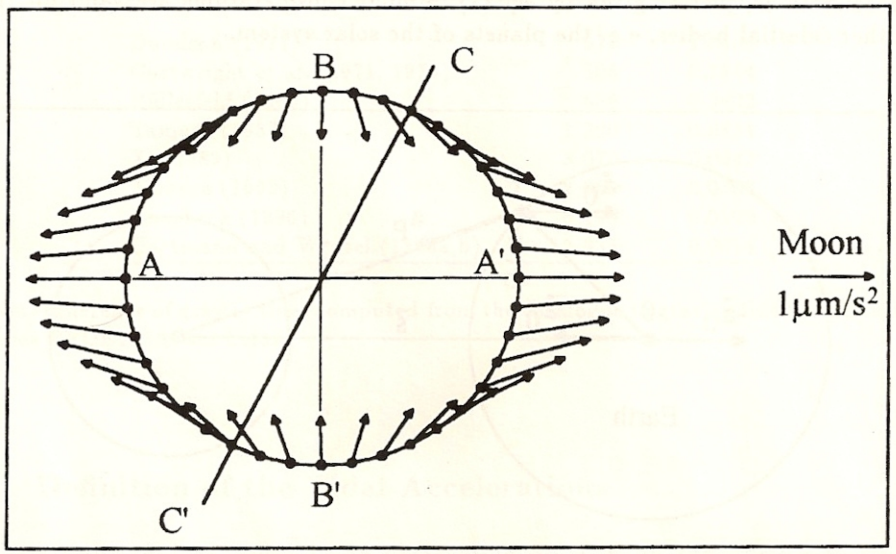


Fig. 2. Tidal accelerations due to the Moon at the surface of the Earth

Acceleration at point P is the difference between homogeneous centrifugal forces and inhomogeneous gravitational force:

$$\vec{b} = \vec{a}_p - \vec{a}_0 = \frac{GM_b}{d^2} \cdot \frac{\vec{d}}{d} - \frac{GM_b \vec{s}}{s^2 \; s}$$

$\vec{d}$: distance between point P and the center of body 2.

Tides

Tidal potential at the Earth's surface:

$$\vec{b} = \nabla V$$

$$\nabla V = \frac{\partial f}{\partial r}\hat{\vec{r}} + \frac{1}{r}\frac{\partial V}{\partial \vartheta}\hat{\vec{\vartheta}} + \frac{1}{r \sin\vartheta}\frac{\partial V}{\partial \varphi}\hat{\vec{\varphi}}$$

$$V = GM_b\left(\frac{1}{d} - \frac{1}{s} - \frac{r\cos\psi}{s^2}\right)$$

ψ: geocentric zenith angle

Expansion into Legendre Polynomials:

$$V = \frac{GM_b}{s} \cdot \sum_{l=2}^{\infty}\left(\frac{r}{s}\right)^l P_l(\cos\psi)$$

Tides

Ratio r/s small:

Body	r/s	$l_{\max}$
Moon	$1.6 \cdot 10^{-2}$	6
Sun	$4.5 \cdot 10^{-5}$	3
Planets		2

Tides

Expressing geocentric zenith angle through

$$P_l(\cos\psi) = \frac{1}{2l+1}\sum_{m=0}^{l} P_{lm}(\cos\vartheta)\cdot P_{lm}(\cos\vartheta_b)\cdot\cos(m\varphi - m\varphi_b)$$

ϑ: geocentric spherical polar distance of the point;
ϑ_b: geocentric spherical polar distance of the second body;
φ: geocentric spherical longitude of the point;
φ_b: geocentric spherical longitude of the second body.
P_{lm}: Fully normalized spherical harmonics

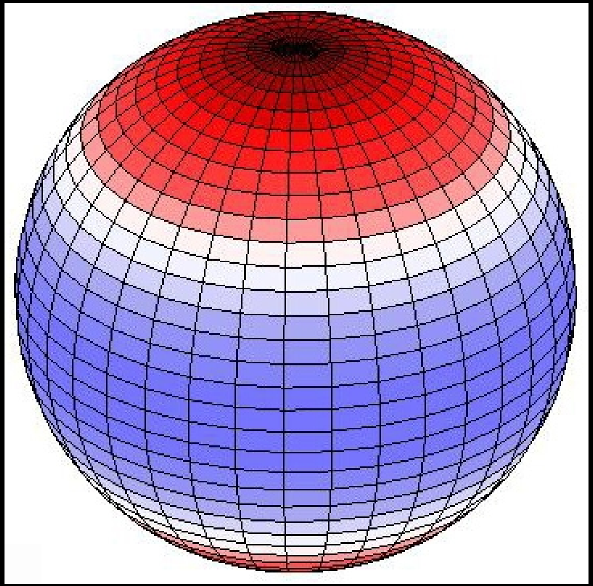
$$V(t) = GM_b \sum_{l=2}^{l_{\max}} \frac{r^l}{s^{l+1}(t)}\frac{1}{2l+1}\sum_{m=0}^{l} P_{lm}(\cos\vartheta)\cdot P_{lm}(\cos\vartheta_b(t))\cdot\cos(m\varphi - m\varphi_b(t))$$

with s, ϑ_b, and φ_b time dependent.

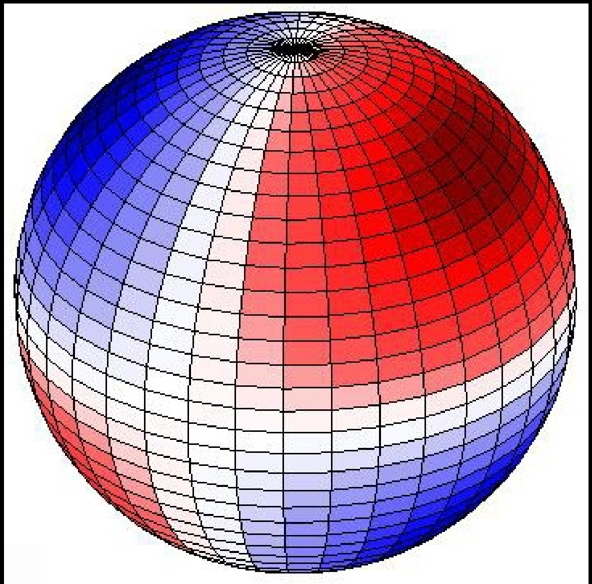
Additional terms due to non-spherical, extended mass of the Earth;
These terms lead to a tidal contribution to the flattening of the Earth.

Tides

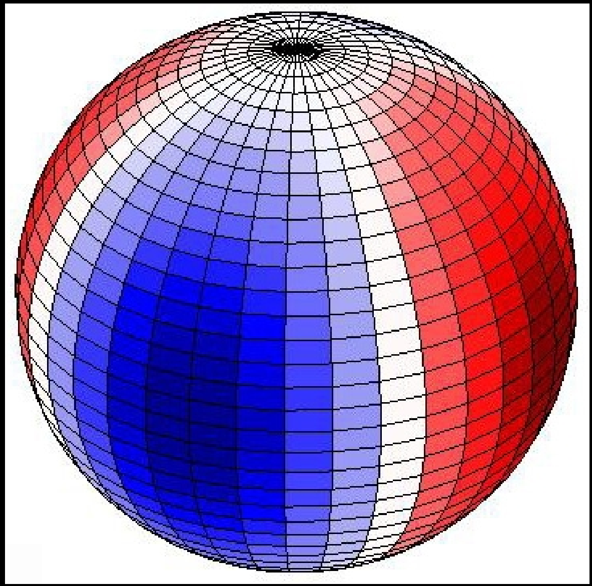
Types of tides:
m = 0: periods 14 days to 18.6 years; long-period constituents
m = 1: periods ~24 hours; diurnal constituents
m = 2: periods ~12 hours; semi-diurnal constituents
m = 3: periods ~ 8 hours; terdiurnal constituents
m = 4: periods ~ 6 hours
m = 5: periods ~ 4.8 hours
m = 6: periods ~ 4 hours



m = 0: zonal,
max. amplitude at poles
periods 14 days to 18.6 years;
long-period constituents



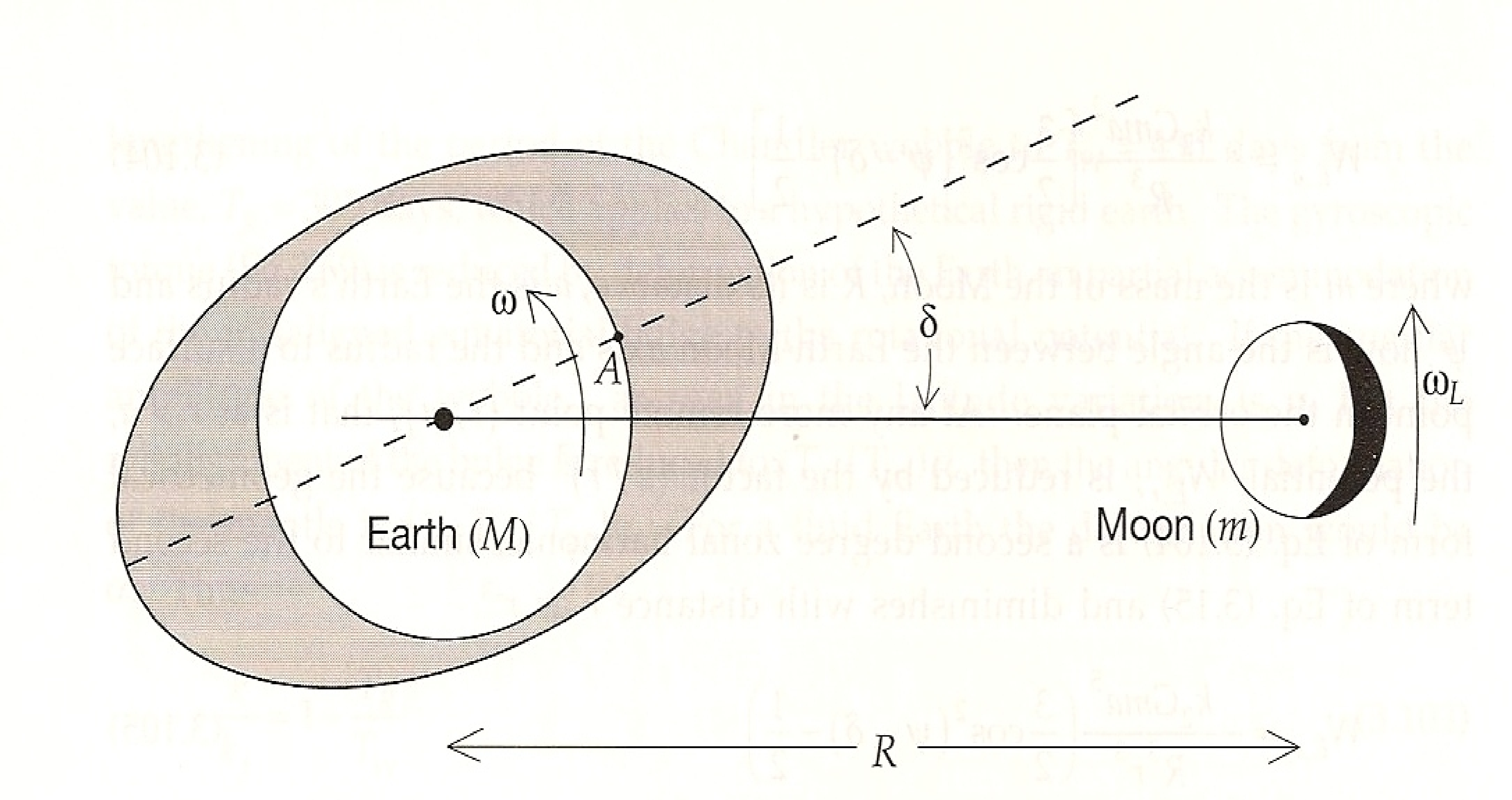
m = 1: tesseral,
max. amplitude at ±45°
periods ~ 1 day;
diurnal constituents



m = 0: sectoral,
max. amplitude at 0°
periods ~0.5 day;
diurnal constituents

Tides

Tidal Friction



Tidal bulge lags behind Moon by about 12 minutes:
- tidal torque
- lag is due to ocean tides, anelasticity in the solid earth

Tides

Computation of tides:
(1) use:

$$V(t) = GM_b \sum_{l=2}^{l_{\max}} \frac{r^l}{s^{l+1}(t)} \frac{1}{2l+1} \sum_{m=0}^{l} P_{lm}(\cos\vartheta) \cdot P_{lm}(\cos\vartheta_b(t)) \cdot \cos(m\varphi - m\varphi_b(t))$$

with s, ϑ_b, and φ_b time dependent.

and ephemerides of Moon, Sun and planets to compute the time dependent parameters.

(2) Use a development into harmonic constituents given in a tidal catalogue

Tides

Tidal potential catalogues

$$V(t) = D \sum_{l=1}^{l_{\max}} \sum_{m=0}^{l} \left(\frac{r}{a}\right)^l \Gamma(\vartheta) \cdot P_{lm}(\cos\vartheta) \cdot \sum_i [C_i^{lm}(t)\cos(\alpha_i(t)) + S_i^{lm}(t)\sin(\alpha_i(t))]$$

D, Γ:	Normalization constants
a:	semi-major axis of the reference ellipsoid
C_i, S_i:	time dependent coefficients of the catalogue; given as $C_i^{lm}(t) = C0_i^{lm} + t \cdot C1_1^{lm}$ and $S_i^{lm}(t) = S0_i^{lm} + t \cdot S1_1^{lm}$.

$$\alpha_i(t) = m \cdot \lambda + \sum_{j=1}^{j_{\max}} k_{ij} \cdot arg_j(t);\; k_{i1} = m$$

k_{ij}:	integer coefficients, given in catalog
$arg_j(t)$:	astronomical arguments, can be computed from ephemerides (polynomials)

Tides

Tidal potential catalogues

catalogue	no. of waves	no. of coeff.	max. degree	truncation $[m^2/s^2]$	ephemerides
Doodson (1921)	378	378	3	$1.0 \cdot 10^{-4}$	BN
Cartwright et al. (1971, 1973)	505	1,010	3	$0.4 \cdot 10^{-4}$	BN
Büllesfeld (1985)	656	656	4	$0.2 \cdot 10^{-4}$	BN
Tamura (1987)	1,200	1,326	4	$0.4 \cdot 10^{-5}$	DE118/LE62
Xi (1989)	2,934	2,934	4	$0.9 \cdot 10^{-6}$	BN
Tamura (1993)	2,060	3,046	4	$0.4 \cdot 10^{-5}$	DE200/LE200
Roosbeck (1996)	6,499	7,202	5	$0.8 \cdot 10^{-7}$	CT/BR
Hartmann & Wenzel (1995a,b)	12,935	19,271	6	$0.1 \cdot 10^{-9}$	DE200/LE200

Naming conventions (examples):

K1: 23.93 hours; declinational wave (both lunar and solar), amplitude 192 mm vertical, 78 mm horiz.
S1: 24.00 hours, solar elliptic wave to K1 (solar), amplitude 1.7 mm vertical, 0.25 horizontal
M2: 12.42 hours, lunar principal (semi-diurnal) wave, amplitude 385 mm vertical, 58 mm horizontal
Mf: 16.47 days, lunar declinational wave, amplitude 40 mm vertical, 6 mm horizontal
Sa: 1 year, solar elliptical wave, amplitude 3 mm vertical, 0.4 mm horizontal

Tides

Solid Earth Tides

Equilibrium tide on rigid Earth:

$$\delta N = \frac{V}{g}$$

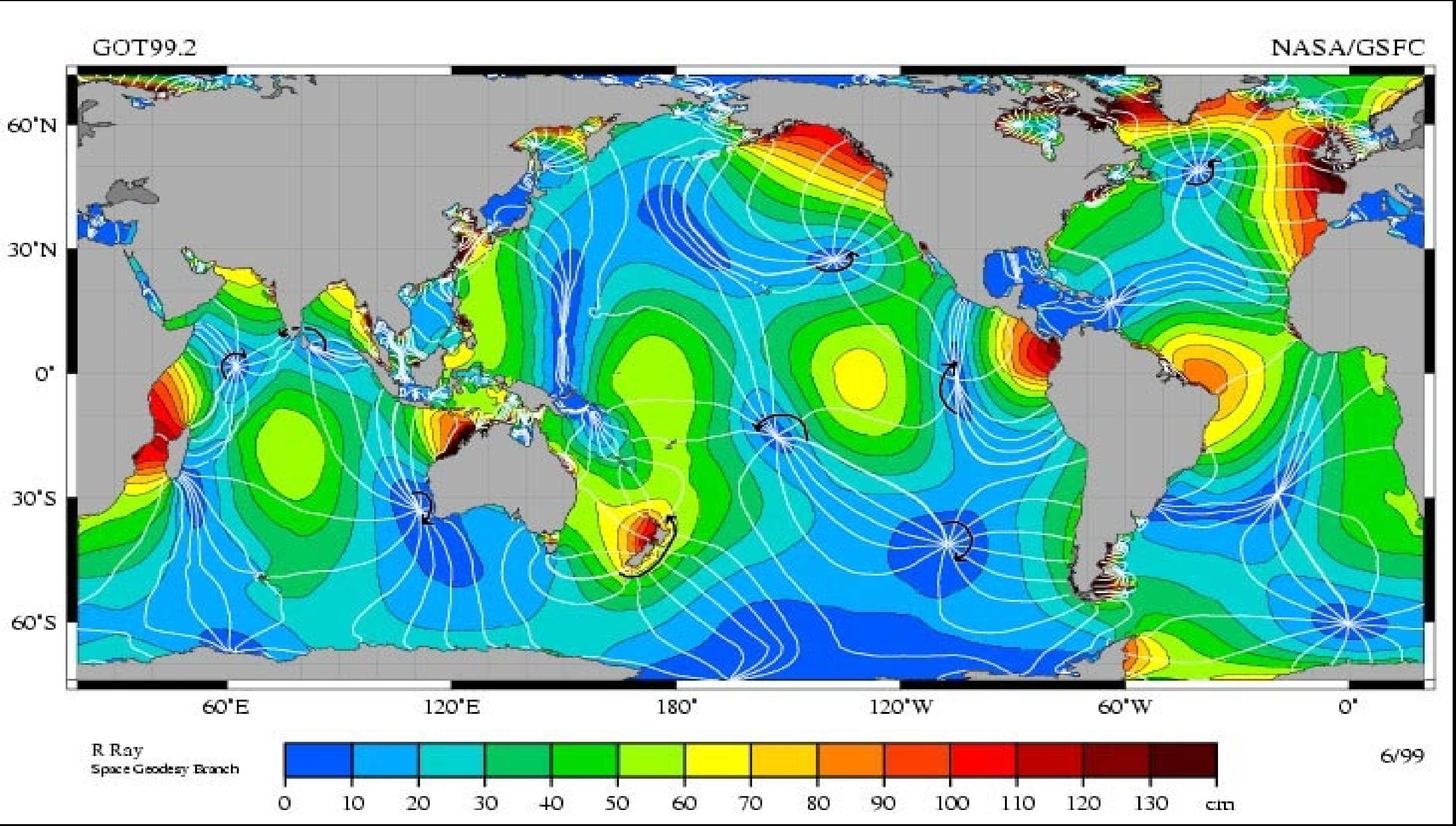
Additional perturbations for elastic Earth expressed through Love/Shida Numbers:

Love/ Shida	$l = 2$	definition
h	0.612	ratio of height of solid body tide to the deforming potential
k	0.303	ratio of additional tidal potential to the deforming potential
l	0.04	ratio of horizontal displacement of the crust to that of the equlibrium tide of a fluid Earth

Love/Shida Numbers depend on *l* and (for non-spherical Earth models) on *m*.

Tides

Ocean tides:
Dominated by the fact that the oceans have eigenperiods in the diurnal to subdiurnal band => dynamic tides



Tides

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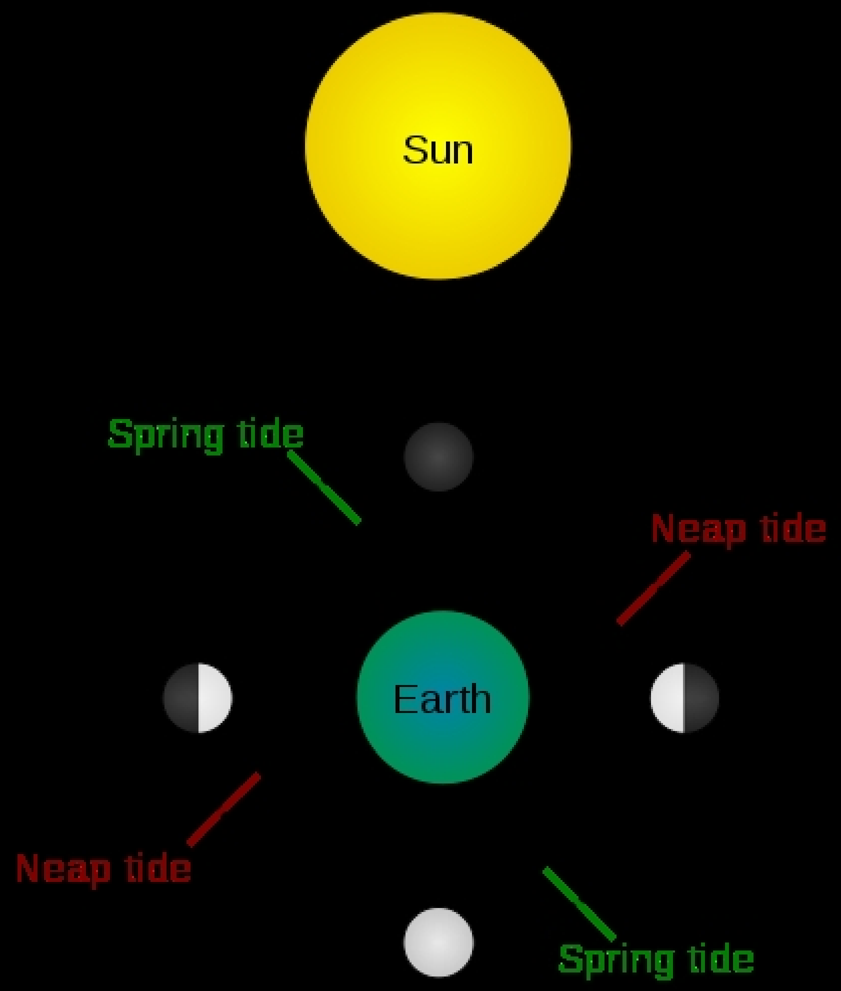
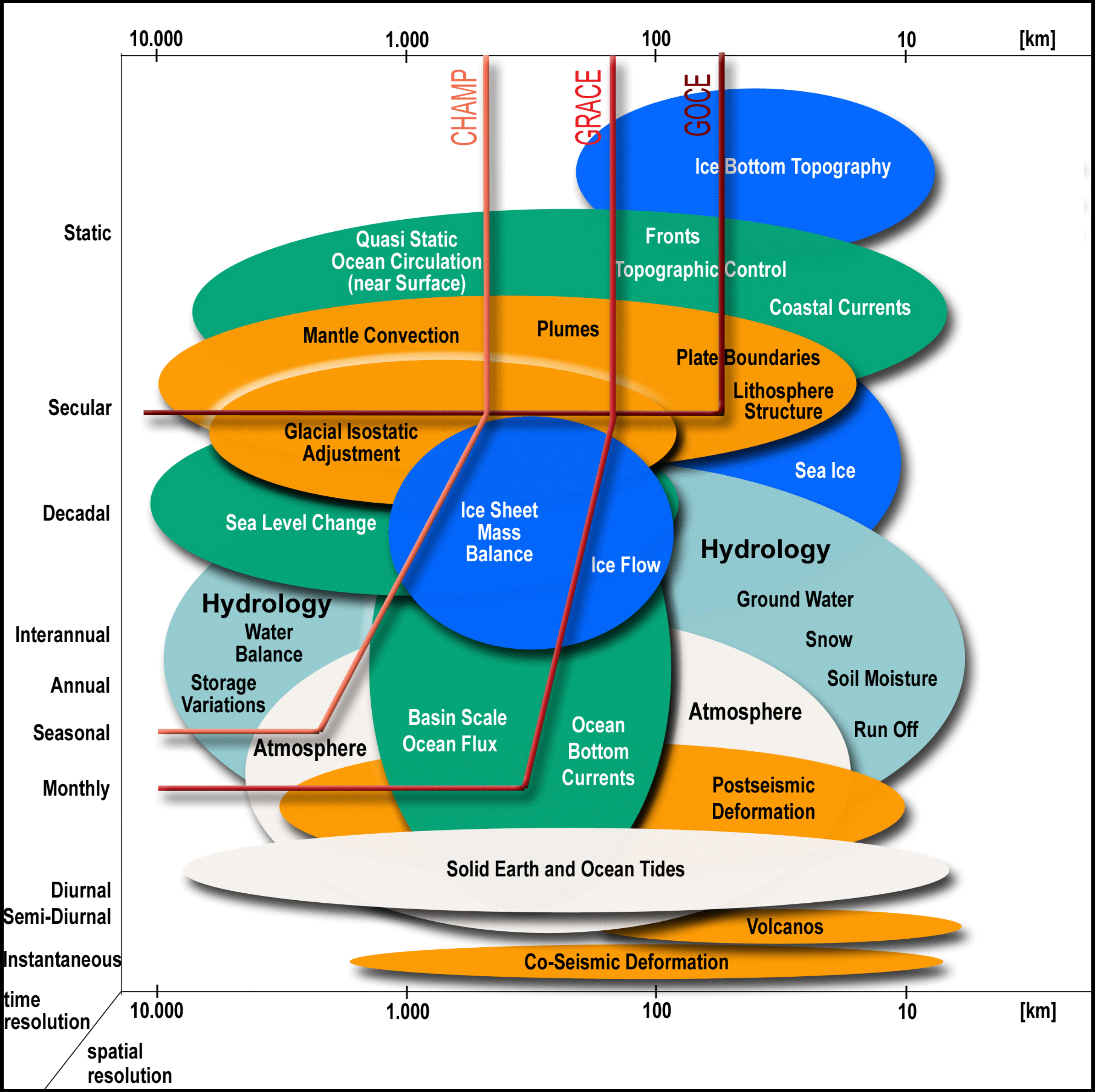
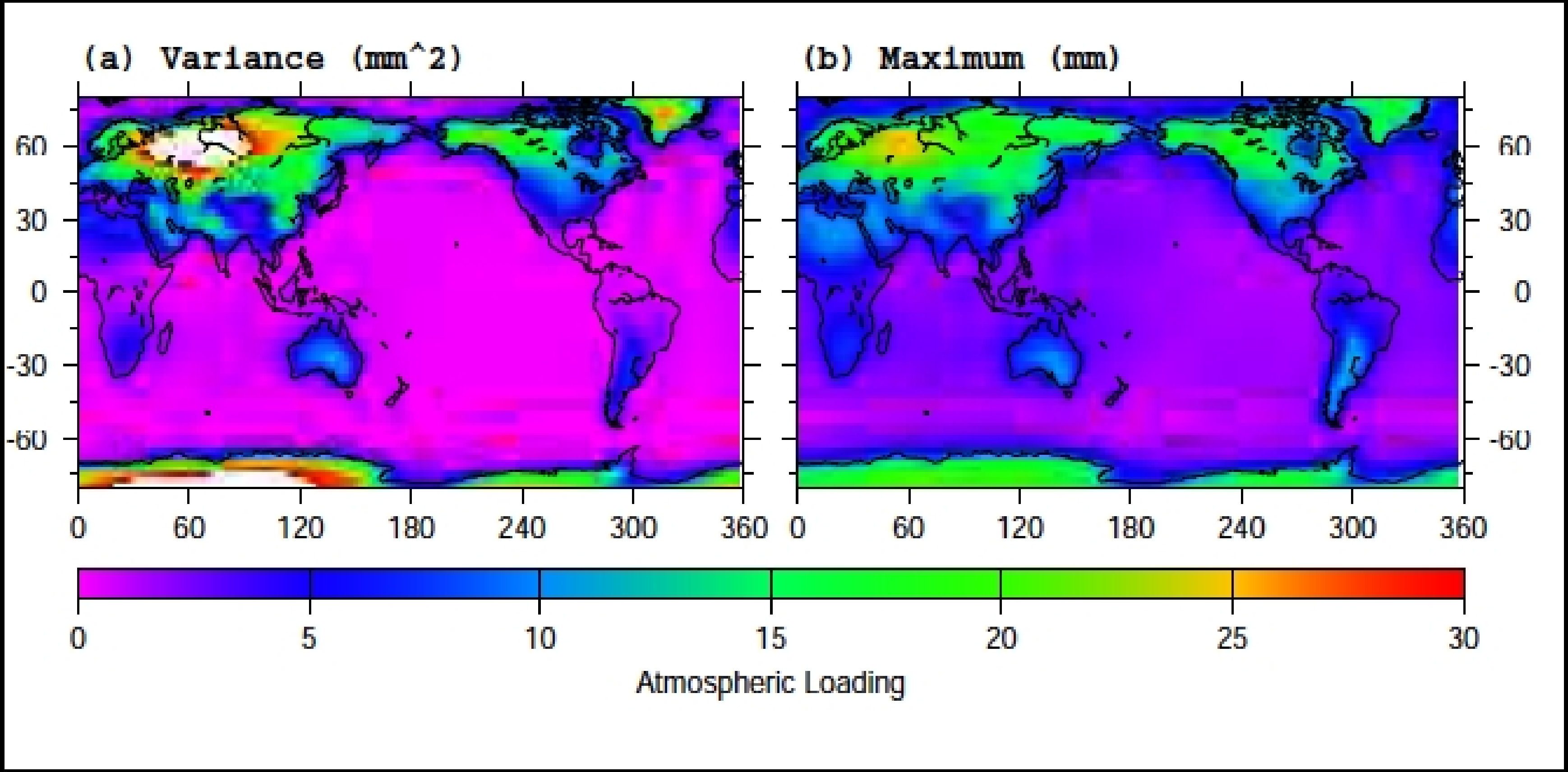


Non-Tidal Variations



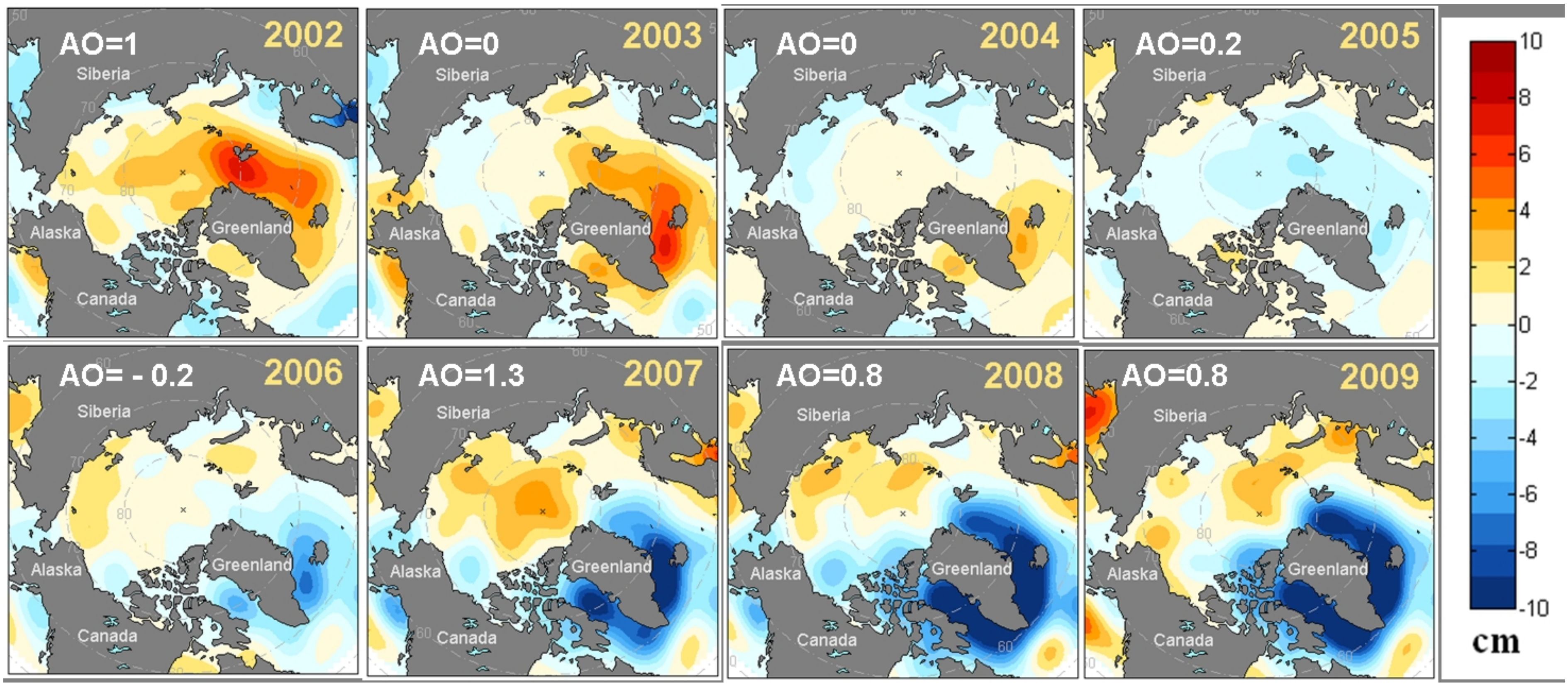
Non-Tidal Variations

Main contributions from:
- atmospheric loading; see an example at http://www.sbl.statkart.no/aboutloading/loading_1999.dr.opt.gif
- non-tidal ocean loading;
- terrestrial hydrosphere;
- land-based glaciers, ice caps and ice sheets.



GRACE Reveals Changes in Arctic Ocean Circulation Patterns

Variations in the Arctic Ocean circulation are associated with clockwise and counterclockwise shifts in the front between salty Atlantic-derived and less salty Pacific-derived upper ocean waters.
Orientation of the front is climatically important because it impacts sea ice transport.

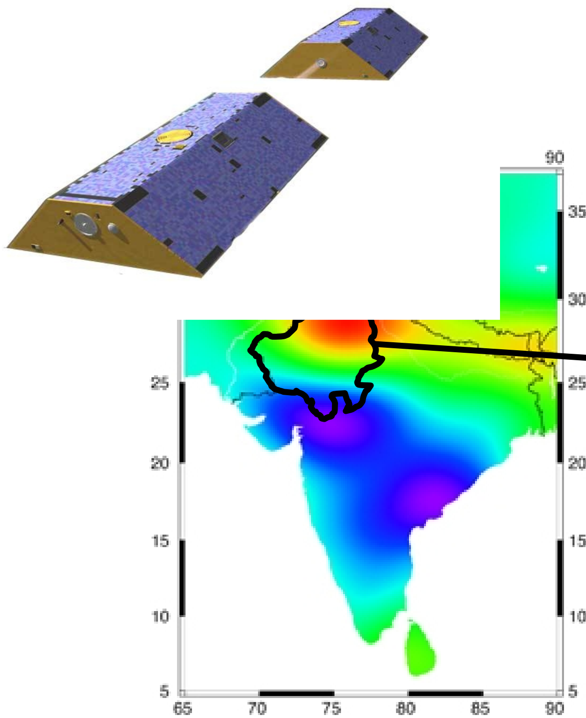


Morrison et al., *GRL*,2007

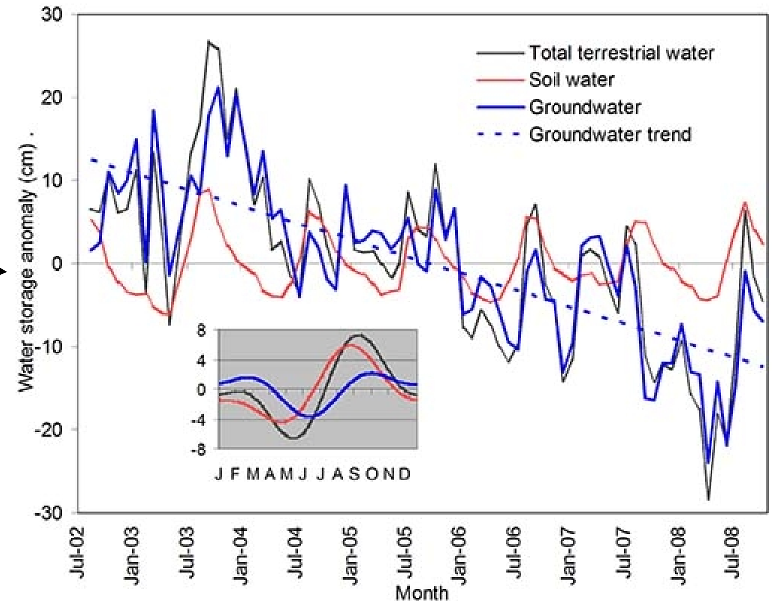
GRACE Quantifies Massive Depletion of Groundwater in NW India

The water table is declining at an average rate of 33 cm/yr

GRACE is unique among Earth observing missions in its ability to monitor variations in all water stored on land, down to the deepest aquifers.



Trends in groundwater storage during 2002-08, with increases in blue and decreases in red. The study region is outlined.



Time series of total water from GRACE, simulated soil water, and estimated groundwater, as equivalent layers of water (cm) averaged over the region. The mean rate of groundwater depletion is 4 cm/yr. Inset: Seasonal cycle.

During the study period, 2002-08, 109 km³ of groundwater was lost from the states of Rajasthan, Punjab, and Haryana; triple the capacity of Lake Mead

Rodell, Velicogna, and Famiglietti, *Nature*, 2009

GRACE Detects Accelerated Ice Mass Loss in Greenland and Antarctica

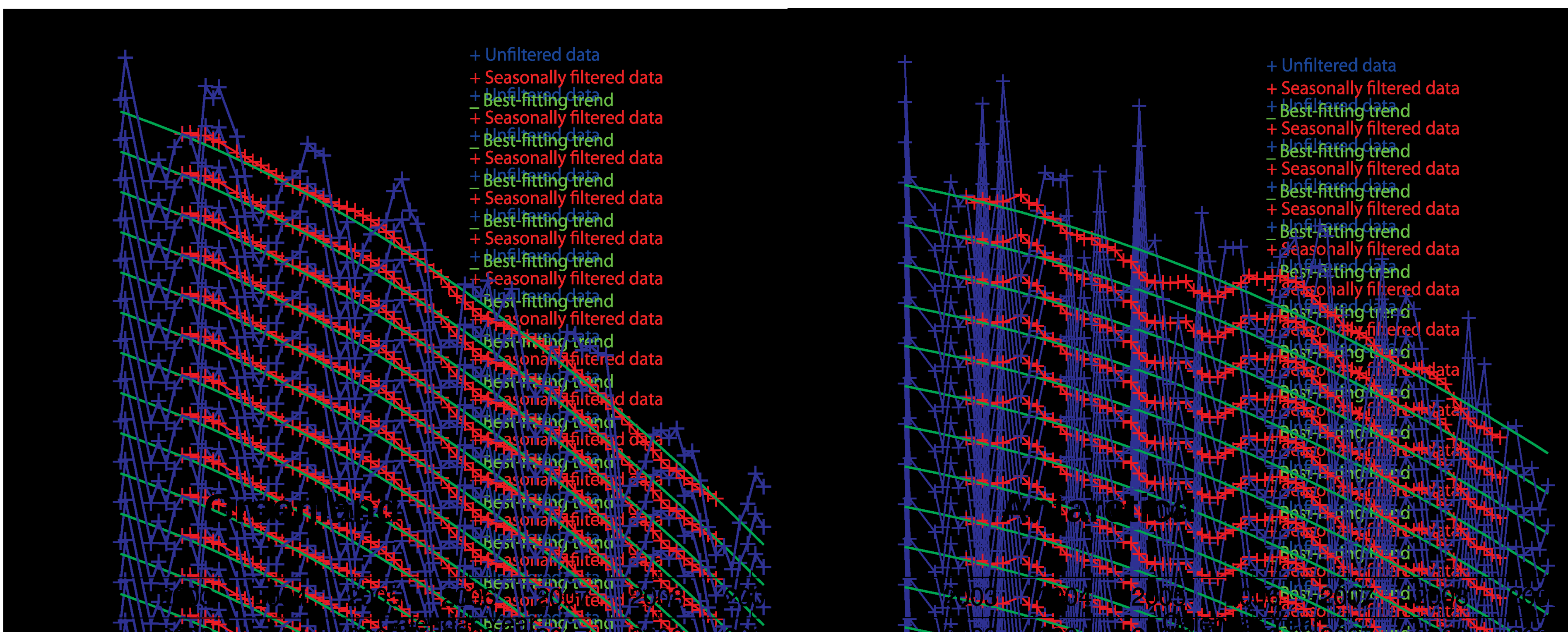
During the period of April 2002 to February 2009 the mass loss of the polar ice sheets was not constant but increased with time, implying that the ice sheets' contribution to sea level rise was increasing.

Greenland:

- mass loss increased from 137 Gt/yr in 2002–2003 to 286 Gt/yr in 2007–2009
- acceleration of -30 ± 11 Gt/yr^2 in 2002–2009.

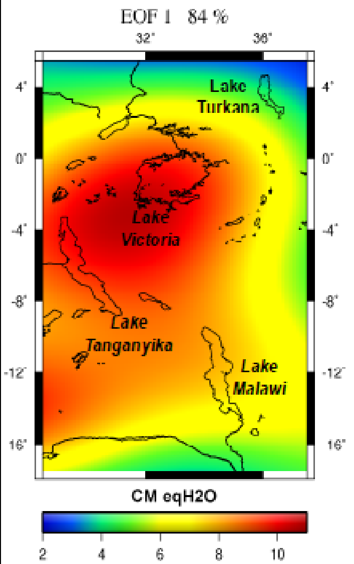
Antarctica:

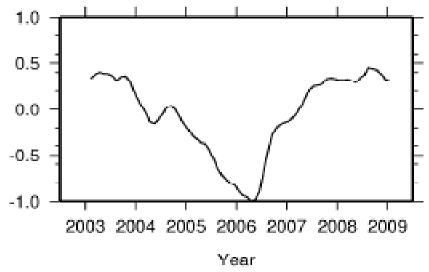
- mass loss increased from 104 Gt/yr in 2002–2006 to 246 Gt/yr in 2006–2009
- acceleration of -26 ± 14 Gt/yr^2 in 2002–2009.



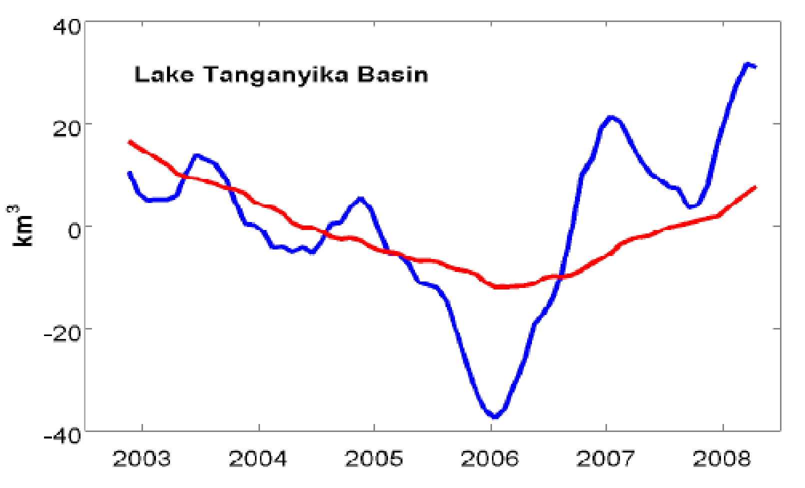
Velicogna, *GRL*,2009

Hydrology:
Seasonal and interannual changes in land-water storage

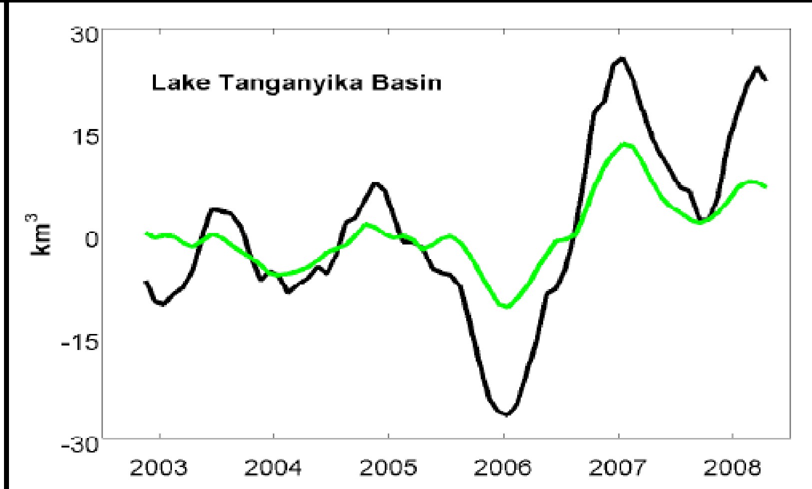




Inter-annual pattern of water storage variations in the East-African lake region based on a statistical analysis (EOF decomposition) of GRACE data.



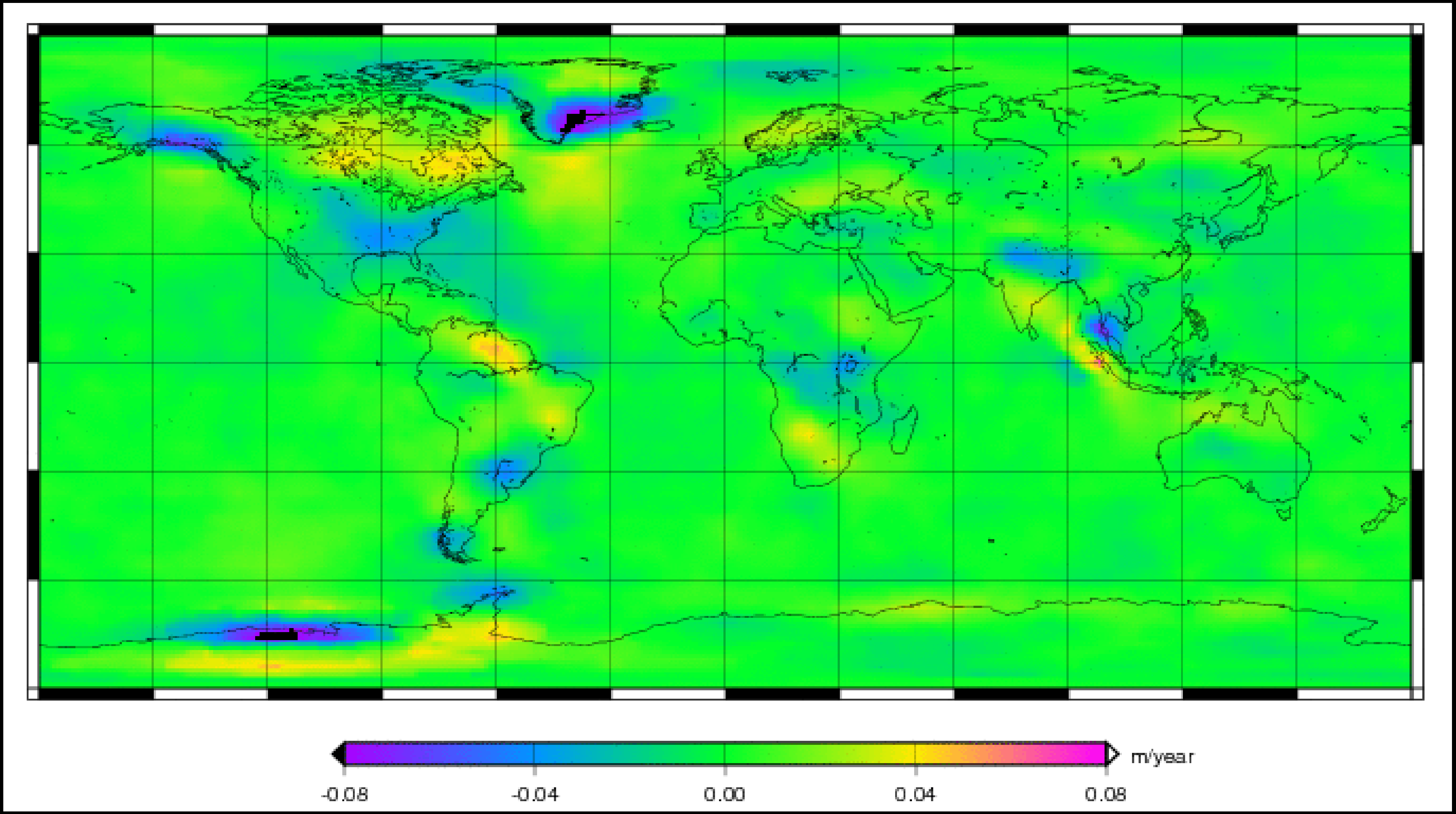
Total water storage from GRACE for the catchment of Lake Tanganyika (blue) and lake water storage from altimetry (red).



GRACE-based subsurface water storage in black (lake storage removed) (black) and simulation results of the global hydrological model WGHM (green).

Becker et al., 2009

Hydrology: Secular trends in Land Water storage



JPL MASCON, secular trends 2003-2007, Watkins, 2008