Viscoelastic deformations of a large, rotating and gravitating body: the basic theory

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0.1 Linearized differential equations for deformations

0.1.1 General setting of the model

The linearized differential equation system for the tidal deformation $\boldsymbol{u}$ and the incremental potential φ is given by

$$\begin{aligned} \rho\ddot{\boldsymbol{u}} &= \rho\left\{\nabla\varphi + (\boldsymbol{u}\cdot\nabla)\nabla(\Phi+\Psi) - 2\boldsymbol{\Omega}\times\dot{\boldsymbol{u}}\right\} + \\ &\quad \nabla(\boldsymbol{u}\cdot\nabla P) - (\nabla\cdot\boldsymbol{u})\nabla P - (\boldsymbol{u}\cdot\nabla)\nabla P + \nabla\cdot\boldsymbol{\sigma}\ , \\ \nabla^2\varphi &= 4\pi G\,\nabla\cdot(\rho\boldsymbol{u})\ , \end{aligned} \tag{1}$$

where $\rho = \rho(\boldsymbol{r})$ denotes the density, Φ is the (negative) gravitational potential

$$\Phi(\boldsymbol{r}) \stackrel{\text{def}}{=} G\int_{\mathcal{V}} \frac{\rho(\boldsymbol{r'})}{|\boldsymbol{r}-\boldsymbol{r'}|}\,\mathrm{d}^3\boldsymbol{r'}\ , \tag{2}$$

with gravitational constant G,

$$\Psi(\boldsymbol{r}) \stackrel{\text{def}}{=} \frac{1}{2}(\Omega^2 r^2 - (\boldsymbol{\Omega}\cdot\boldsymbol{r})^2) = \frac{1}{2}[\boldsymbol{\Omega}\times\boldsymbol{r}]^2 \tag{3}$$

with $\Omega = |\boldsymbol{\Omega}|$, $r = |\boldsymbol{r}|$ denotes the (negative) rotational potential for angular velocity vector $\boldsymbol{\Omega}$, P the hydrostatic pressure and $\boldsymbol{\sigma}$ is the incremental pseudo-stress tensor with components

$$\sigma_{ij} = \lambda\partial_k u_k\delta_{ij} + \mu(\partial_i u_j + \partial_j u_i)\ , \tag{4}$$

when the material is assumed to be isotropic. Here, λ and μ denote the Lamé parameters.

Since in hydrostatic equilibrium the pressure P must satisfy

$$\nabla P = \rho\nabla(\Phi+\Psi)\ , \tag{5}$$

it follows $\nabla\rho\times\nabla(\Phi+\Psi) = 0$ and the first equation (1) may be simplified to give

$$\rho\ddot{\boldsymbol{u}} = \rho\left\{\nabla(\varphi + \boldsymbol{u}\cdot\nabla(\Phi+\Psi)) - (\nabla\cdot\boldsymbol{u})\nabla(\Phi+\Psi) - 2\boldsymbol{\Omega}\times\dot{\boldsymbol{u}}\right\} + \nabla\cdot\boldsymbol{\sigma}\ . \tag{6}$$

Equation system (6) has to be suplemented by interior boundary conditions

$$[\boldsymbol{n}\cdot\boldsymbol{\sigma}]_-^+ = 0\ , \tag{0.7a}$$

$$\left\{\begin{aligned} [\boldsymbol{u}]_-^+ &= 0 \quad \text{if welded} \\ [\boldsymbol{n}\cdot\boldsymbol{u}]_-^+ &= 0 \quad \text{if frictionless} \end{aligned}\right\}\ , \tag{0.7b}$$

$$[\varphi]_-^+ = 0\ , \tag{0.7c}$$

$$[\boldsymbol{n}\cdot(\nabla\varphi - 4\pi G\rho\boldsymbol{u})]_-^+ = 0\ , \tag{0.7d}$$

and surface boundary conditions

$$\boldsymbol{n}\cdot\boldsymbol{\sigma}\big|_- = \boldsymbol{F}\ , \tag{0.8a}$$

$$\varphi\big|_- = \varphi\big|_+\ , \tag{0.8b}$$

$$\boldsymbol{n}\cdot(\nabla\varphi - 4\pi G\rho\boldsymbol{u})\big|_- = \boldsymbol{n}\cdot\nabla\varphi\big|_+ + 4\pi G\,\tau\ , \tag{0.8c}$$

where $\boldsymbol{n}$ denotes the exterior normal to the boundary, $\boldsymbol{F}$ is the external force per area acting on the surface, and τ is the surface mass density. If one is studying body tides, one has $\boldsymbol{F} = \boldsymbol{0}$ and $\tau = 0$. However, for loading tides $\boldsymbol{F}$ and τ are the sources, which generate the tides.

The last equation in (1) together with the boundary conditions (0.8b) and (0.8c) may be solved by the methods of potential theory to give

$$\varphi(\boldsymbol{r}) = -G\int_{\mathcal{V}} \frac{\nabla\cdot(\rho(\boldsymbol{r'})\boldsymbol{u}(\boldsymbol{r'}))}{|\boldsymbol{r}-\boldsymbol{r'}|}\,\mathrm{d}^3\boldsymbol{r'} + G\int_{\partial\mathcal{V}} \frac{\tau(\boldsymbol{r'}) + \rho(\boldsymbol{r'})\,(\boldsymbol{u}(\boldsymbol{r'})\cdot\boldsymbol{n'})}{|\boldsymbol{r}-\boldsymbol{r'}|}\mathrm{d}^2\boldsymbol{r'} + \varphi^{\mathrm{P}}(\boldsymbol{r}) \tag{9}$$

where φ^{P} denotes the tide generating primary potential which is smooth and satisfies the Laplace equation $\nabla^2\varphi^{\mathrm{P}} = 0$ inside and at the surface of the earth.

0.1.2 First order differential equation system

We take the undisturbed system to be spherically symmetric ($\rho = \rho(r)$) and non-rotating. Then equation (1) simplifies further to

$$\begin{aligned} \rho\ddot{\boldsymbol{u}} &= \rho\{\nabla[\varphi - g(\boldsymbol{u}\cdot\boldsymbol{e}_r)] + g(\nabla\cdot\boldsymbol{u})\boldsymbol{e}_r\} + \nabla\cdot\boldsymbol{\sigma} \;, \\ \nabla^2\varphi &= 4\pi G\,\nabla\cdot(\rho\boldsymbol{u}) \;, \end{aligned} \tag{10}$$

where $g = -\mathrm{d}\Phi/\mathrm{d}r$ is the absolut value of the gravitational acceleration. Equation (9) may be evaluated for the spherically symmetric case. It is, however, more convenient to integrate the whole differential equation system (10) directly without making use of (9). This system allows for two types of solutions, toroidal and spheroidal solutions. The toroidal solutions obey $\nabla\cdot\boldsymbol{u} = \nabla\cdot(\rho\boldsymbol{u}) = 0$ and are obtained from the ansatz

$$\begin{aligned} \boldsymbol{u} &= W_n(r)\,\boldsymbol{e}_r \times \nabla_1 Y_{nm}(\vartheta,\chi)\exp(i\omega t) \;, \\ \varphi &= 0 \;, \end{aligned} \tag{11}$$

where we have agreed to set $\nabla_1 \stackrel{\text{def}}{=} r[\nabla - \boldsymbol{e}_r(\boldsymbol{e}_r\cdot\nabla)]$ and $Y_{nm}(\vartheta,\chi)$ is the spherical harmonic of degree n and order m given by

$$\begin{aligned} Y_{nm}(\vartheta,\chi) &= \sqrt{\tfrac{2n+1}{4\pi}\tfrac{(n-m)!}{(n+m)!}}\,P_n^m(\cos\vartheta)\,e^{im\chi} \;, \\ Y_{n\,-m}(\vartheta,\chi) &= (-1)^m\,Y^*_{nm}(\vartheta,\chi) \;, \qquad \text{with} \\ P_n^m(\cos\vartheta) &= (-1)^m\tfrac{(1-\cos^2\vartheta)^{m/2}}{2^{\,n}\,n!}\tfrac{d^{\,n+m}}{d(\cos\vartheta)^{\,n+m}}(\cos^2\vartheta - 1)^n \;. \end{aligned} \tag{12}$$

For this ansatz we have

$$\begin{aligned} \nabla\cdot\boldsymbol{\sigma} \;=\; & \left\{\frac{\mathrm{d}}{\mathrm{d}r}\left[\mu\left(W_n'(r) - \frac{1}{r}W_n(r)\right)\right]\right. \\ & \left. +\frac{\mu}{r}\left(3W_n'(r) - \frac{1+n(n+1)}{r}W_n(r)\right)\right\}\boldsymbol{e}_r \times \nabla_1 Y_{nm}(\vartheta,\chi)\exp(i\omega t) \end{aligned} \tag{13}$$

with $W_n'(r) \stackrel{\text{def}}{=} \mathrm{d}W_n(r)/\mathrm{d}r$. Introducing the "solution vector"

$$\mathbf{y}_\mathrm{T} \stackrel{\text{def}}{=} \sqrt{r}\begin{pmatrix} W_n \\ \mu(rW_n' - W_n)/(n+1) \end{pmatrix} \;, \tag{14}$$

equation (10) for toroidal solutions reads

$$r\frac{\mathrm{d}\mathbf{y}_\mathrm{T}}{\mathrm{d}r} = \mathbf{A}_\mathrm{T}\mathbf{y}_\mathrm{T} \;, \tag{15}$$

where the matrix $\mathbf{A}_\mathrm{T}$ is defined as

$$\mathbf{A}_\mathrm{T} = \begin{pmatrix} \frac{3}{2} & \frac{n+1}{\mu} \\ \frac{(n-1)(n+2)\mu - \rho\omega^2 r^2}{n+1} & -\frac{3}{2} \end{pmatrix} \;. \tag{16}$$

Spheroidal solutions are obtained from the ansatz

$$\begin{aligned} \boldsymbol{u} &= \{H_n(r)\boldsymbol{e}_r + T_n(r)\nabla_1\}Y_{nm}(\vartheta,\chi)\exp(i\omega t) \;, \\ \varphi &= R_n(r)Y_{nm}(\vartheta,\chi)\exp(i\omega t) \end{aligned} \tag{17}$$

and yuield

$$\begin{aligned} &\nabla\cdot\boldsymbol{\sigma} = \\ &\left\{\frac{\mathrm{d}}{\mathrm{d}r}\left[\lambda\left(H_n'(r) + \frac{2}{r}H_n(r) - \frac{n(n+1)}{r}T_n(r)\right) + 2\mu H_n'(r)\right] + \frac{4\mu}{r}\left(H_n'(r) - \frac{1}{r}H_n(r)\right)\right. \\ &\left. -\frac{n(n+1)\mu}{r}\left(T_n'(r) - \frac{3}{r}T_n(r) + \frac{1}{r}H_n(r)\right)\right\}\boldsymbol{e}_r Y_{nm}(\vartheta,\chi)\exp(i\omega t) + \\ &\left\{\frac{\lambda}{r}\left(H_n'(r) + \frac{2}{r}H_n(r) - \frac{n(n+1)}{r}T_n(r)\right) + \frac{\mathrm{d}}{\mathrm{d}r}\left[\mu\left(T_n'(r) - \frac{1}{r}T_n(r) + \frac{1}{r}H_n(r)\right)\right]\right. \\ &\left. +\frac{\mu}{r}\left(3T_n'(r) - \frac{1+2n(n+1)}{r}T_n(r) + \frac{5}{r}H_n(r)\right)\right\}\nabla_1 Y_{nm}(\vartheta,\chi)\exp(i\omega t) \;. \end{aligned} \tag{18}$$

For the case $n > 0$ we introduce the "solution vector"

$$\mathbf{y}_\mathrm{S} \stackrel{\text{def}}{=} \sqrt{r}\begin{pmatrix} H_n \\ [(\lambda+2\mu)rH_n' + 2\lambda H_n]/(n+1) - n\lambda T_n \\ nT_n \\ \mu(rT_n' - T_n + H_n) \\ R_n \\ rR_n' + (n+1)R_n - 4\pi G\rho r H_n \end{pmatrix} \tag{19}$$

and arive at an equation

$$r\frac{\mathrm{d}\mathbf{y}_{\mathrm{S}}}{\mathrm{d}r} = \mathbf{A}_{\mathrm{S}}\mathbf{y}_{\mathrm{S}} \tag{20}$$

with a matrix[1]

$$\mathbf{A}_{\mathrm{S}} = \begin{pmatrix} \frac{-3\lambda+2\mu}{2(\lambda+2\mu)} & \frac{n+1}{\lambda+2\mu} & \frac{(n+1)\lambda}{\lambda+2\mu} & 0 & 0 & 0 \\ \frac{4\mu(3\lambda+2\mu)}{(n+1)(\lambda+2\mu)} - \frac{4\rho g r+\rho\omega^2 r^2}{n+1} & \frac{3\lambda-2\mu}{2(\lambda+2\mu)} & \rho g r - \frac{2\mu(3\lambda+2\mu)}{\lambda+2\mu} & n & \rho r & \frac{-\rho r}{n+1} \\ -n & 0 & \frac{3}{2} & \frac{n}{\mu} & 0 & 0 \\ \rho g r - \frac{2\mu(3\lambda+2\mu)}{\lambda+2\mu} & \frac{-(n+1)\lambda}{\lambda+2\mu} & \frac{4\mu(n+1)(\lambda+\mu)}{\lambda+2\mu} - \frac{2\mu+\rho\omega^2 r^2}{n} & -\frac{3}{2} & -\rho r & 0 \\ 4\pi G\rho r & 0 & 0 & 0 & -\left(n+\frac{1}{2}\right) & 1 \\ 4\pi G\rho(n+1)r & 0 & -4\pi G\rho(n+1)r & 0 & 0 & n+\frac{1}{2} \end{pmatrix} . \tag{21}$$

These equations have first been given by Wiggins **?**. For $n = 0$ the term $T_0\nabla_1 Y_{00}$ is identical zero and the incremental potential φ is equal to the primary potential φ^{P} outside the earth and $\nabla\varphi = 4\pi G\rho\boldsymbol{u} + const.$ inside the earth since it must be singularity free there. We may therefore introduce the simplified "solution vector"

$$\mathbf{y}_0 \stackrel{\mathrm{def}}{=} \begin{pmatrix} \sqrt{r}H_0 \\ \sqrt{r}[(\lambda+2\mu)rH_0' + 2\lambda H_0] \\ R_0 \end{pmatrix} , \tag{22}$$

which yields the equation

$$r\frac{\mathrm{d}\mathbf{y}_0}{\mathrm{d}r} = \mathbf{A}_0\mathbf{y}_0 \tag{23}$$

with

$$\mathbf{A}_0 = \begin{pmatrix} \frac{-3\lambda+2\mu}{2(\lambda+2\mu)} & \frac{1}{\lambda+2\mu} & 0 \\ \frac{4\mu(3\lambda+2\mu)}{\lambda+2\mu} - 4\rho g r - \rho\omega^2 r^2 & \frac{3\lambda-2\mu}{2(\lambda+2\mu)} & 0 \\ 4\pi G\rho\sqrt{r} & 0 & 0 \end{pmatrix} . \tag{24}$$

The case of a liquid layer has to be considered seperately[2]. For a liquid compressible layer the Adams-Williamson condition

$$\frac{\mathrm{d}\rho}{\mathrm{d}r} + \rho^2\frac{g}{\lambda} = 0 \tag{25}$$

must be satisfied in a state of thermodynamic equilibrium. Since for $\mu = 0$ the Lamé parameter λ is identical with the compression modulus which in turn is nothing else but the adiabatic compressibility

$$\lambda = \rho\left(\frac{\partial P}{\partial\rho}\right)_S , \tag{26}$$

where the subscript S denotes constancy of entropy, (25) may be derived from the mechanical equilibrium condition (5) for the static pressure P by the assumption that density variations in the liquid layer are isentropic.

For $\mu = 0$ the r-component of the first equation (10) together with the ansatz (17) yields

$$\begin{aligned} -\rho\omega^2 H_n \;=\; & \rho\frac{\mathrm{d}}{\mathrm{d}r}(R_n - gH_n) + \rho g\left\{H_n' + \frac{2}{r}H_n - \frac{n(n+1)}{r}T_n\right\} + \\ & \frac{\mathrm{d}}{\mathrm{d}r}\left(\lambda\left\{H_n' + \frac{2}{r}H_n - \frac{n(n+1)}{r}T_n\right\}\right) , \end{aligned} \tag{27}$$

while the ϑ- or χ-components of the first equation (10) yield

$$-\rho\omega^2 rT_n = \rho(R_n - gH_n) + \lambda\left\{H_n' + \frac{2}{r}H_n - \frac{n(n+1)}{r}T_n\right\} . \tag{28}$$

By differentiating this equation with respect to r, adding $\rho g/\lambda$ times this equation, substituting $\mathrm{d}\rho/\mathrm{d}r$ from equation (25) and substracting equation (27) we obtain

$$T_n' = \frac{H_n - T_n}{r} . \tag{29}$$

With the ansatz (17) and the "solution vector"

$$\mathbf{y}_{\mathrm{L}} = \sqrt{r}\begin{pmatrix} 4\pi G\rho rH_n \\ nrT_n \\ nR_n \\ rR_n' + (n+1)R_n - 4\pi G\rho rH_n \end{pmatrix} , \tag{30}$$

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[1] The matrices $\mathbf{A}_{\mathrm{S}}$ and $\mathbf{A}_{\mathrm{L}}$ given here are the $(n+1)$-fold of the matrices $\widetilde{\mathbf{A}}$ and $\widetilde{\mathbf{B}}$ respectively given in Appendix A of **?** corrected for a mistake made there in $\widetilde{\mathrm{A}}_{11}$.

[2] Here, a layer is meant to be liquid if the shear stress vanishes ($\mu = 0$) throughout the layer. This is not fully equivalent to a liquid described by hydrodynamic equations since the deformations $\boldsymbol{u}$ were still supposed to remain small for all times.

we then get the equation

$$r\frac{\mathrm{d}\mathbf{y}_{\mathrm{L}}}{\mathrm{d}r} = \mathbf{A}_{\mathrm{L}}\mathbf{y}_{\mathrm{L}} \tag{31}$$

with a matrix

$$\mathbf{A}_{\mathrm{L}} = \begin{pmatrix} -\frac{1}{2} & 4\pi G\rho\left(n+1-\frac{\rho\omega^2r^2}{n\lambda}\right) & \frac{-4\pi G\rho^2r^2}{n\lambda} & 0 \\ \frac{n}{4\pi G\rho} & \frac{1}{2} & 0 & 0 \\ n & 0 & -\left(n+\frac{1}{2}\right) & n \\ n+1 & -4\pi G\rho(n+1) & 0 & n+\frac{1}{2} \end{pmatrix} . \tag{32}$$

However, for $n = 0$ this ansatz does not work. In this case we have to take the ansatz (22) and get the equation (23) for the liquid layer as well.

The boundary conditions (0.7) for a frictionless boundary between a solid and a liquid layer may be given explicitly as follows:

$$\left\{\begin{array}{l} \mathbf{y}_{\mathrm{S}1} = \frac{1}{4\pi G\rho_{\mathrm{L}}r}\mathbf{y}_{\mathrm{L}1} \ , \\ \mathbf{y}_{\mathrm{S}2} = \frac{g}{4\pi G(n+1)}\mathbf{y}_{\mathrm{L}1} - \frac{\rho_{\mathrm{L}}\omega^2 r}{n(n+1)}\mathbf{y}_{\mathrm{L}2} - \frac{\rho_{\mathrm{L}}r}{n(n+1)}\mathbf{y}_{\mathrm{L}3} \ , \\ \mathbf{y}_{\mathrm{S}4} = 0 \ , \\ \mathbf{y}_{\mathrm{S}5} = \frac{1}{n}\mathbf{y}_{\mathrm{L}3} \ , \\ \mathbf{y}_{\mathrm{S}6} = \mathbf{y}_{\mathrm{L}4} \ . \end{array}\right\} \tag{33}$$

0.1.3 Energy integrals

Here we shall take the potential energy of a deformation, as was given by Pekeris and Jarosch, Jobert, Backus, Backus and Gilbert and by Dahlen **?????**, for the spherically symmetric model and evaluate it to a simple integral over the radial distance. The energy released directly in a load will be given also. The work done by the displacement field against pressure and gravity of the undeformed system is

$$\begin{aligned} \mathcal{P} &= -\frac{1}{2}\int_{\mathcal{V}} \boldsymbol{u}^* \cdot \{\nabla(\boldsymbol{u}\cdot\nabla P) - (\nabla\cdot\boldsymbol{u})\nabla P - (\boldsymbol{u}\cdot\nabla)\nabla P + \rho(\boldsymbol{u}\cdot\nabla)\nabla\Phi\}\ \mathrm{d}^3\boldsymbol{r} \\ &= -\frac{1}{2}\int_{\mathcal{V}} \rho\boldsymbol{u}^* \cdot \{g\boldsymbol{e}_r(\nabla\cdot\boldsymbol{u}) - \nabla\left(g(\boldsymbol{u}\cdot\boldsymbol{e}_r)\right)\}\ \mathrm{d}^3\boldsymbol{r} \ , \end{aligned} \tag{34}$$

where $\mathcal{V}$ denotes the volume of the earth and P is the pressure, the gradient of which is $\nabla P = \rho\nabla\Phi = -\rho g\boldsymbol{e}_r$ by (5). The energy stored in the incremental potential is given by

$$\begin{aligned} \mathcal{G} &= -\frac{1}{2}\int_{\mathcal{V}} \rho\boldsymbol{u}^* \cdot \nabla\varphi\,\mathrm{d}^3\boldsymbol{r} - \frac{1}{2}\int_{\partial\mathcal{V}} \varphi\tau^*\mathrm{d}^2\boldsymbol{r} \\ &= \begin{cases} -\frac{1}{8\pi G}\int_{\mathrm{IR}^3} \nabla\varphi^*\cdot\nabla\varphi\,\mathrm{d}^3\boldsymbol{r} & \text{for loading tides} \\ -\frac{1}{8\pi G}\int_{\mathcal{V}} \nabla\varphi^*\cdot\nabla\varphi\,\mathrm{d}^3\boldsymbol{r} + \frac{1}{8\pi G}\int_{\partial_+\mathcal{V}} \varphi\boldsymbol{e}_r\cdot\nabla\varphi^*\mathrm{d}^2\boldsymbol{r} & \text{for body tides} \end{cases} , \end{aligned} \tag{35}$$

where the upper line in the second expression is only valid for loading tides because the gradient of the incremental potential only then decreases sufficiently fast at infinity. This part of the potential energy may be further decomposed into two parts, $\mathcal{G} = \mathcal{G}^{\mathrm{P}} + \mathcal{G}^{\mathrm{S}}$, according to the decomposition $\varphi = \varphi^{\mathrm{P}} + \varphi^{\mathrm{S}}$ of the potential into a primary and secondary one. In this decomposition, the primary potential φ^{P} is due to other celestial bodies or to the masses on the surface and the secondary potential φ^{S} is due to the mass redistribution within the earth. These parts of the energy are

$$\begin{aligned} \mathcal{G}^{\mathrm{P}} &= -\frac{1}{2}\int_{\mathcal{V}} \rho\boldsymbol{u}^* \cdot \nabla\varphi^{\mathrm{P}}\,\mathrm{d}^3\boldsymbol{r} - \frac{1}{2}\int_{\partial\mathcal{V}} \varphi\tau^*\mathrm{d}^2\boldsymbol{r} \\ &= \begin{cases} -\frac{1}{8\pi G}\int_{\mathrm{IR}^3} \nabla\varphi^*\cdot\nabla\varphi^{\mathrm{P}}\,\mathrm{d}^3\boldsymbol{r} - \frac{1}{2}\int_{\partial\mathcal{V}} \varphi^{\mathrm{S}}\tau^*\mathrm{d}^2\boldsymbol{r} & \text{for loading tides} \\ -\frac{1}{8\pi G}\int_{\mathcal{V}} \nabla\varphi^*\cdot\nabla\varphi^{\mathrm{P}}\,\mathrm{d}^3\boldsymbol{r} + \frac{1}{8\pi G}\int_{\partial_+\mathcal{V}} \varphi^{\mathrm{P}}\boldsymbol{e}_r\cdot\nabla\varphi^*\mathrm{d}^2\boldsymbol{r} & \text{for body tides} \end{cases} \end{aligned} \tag{36}$$

and

$$\begin{aligned} \mathcal{G}^{\mathrm{S}} &= -\frac{1}{2}\int_{\mathcal{V}} \rho\boldsymbol{u}^* \cdot \nabla\varphi^{\mathrm{S}}\,\mathrm{d}^3\boldsymbol{r} \\ &= \begin{cases} -\frac{1}{8\pi G}\int_{\mathrm{IR}^3} \nabla\varphi^*\cdot\nabla\varphi^{\mathrm{S}}\,\mathrm{d}^3\boldsymbol{r} + \frac{1}{2}\int_{\partial\mathcal{V}} \varphi^{\mathrm{S}}\tau^*\mathrm{d}^2\boldsymbol{r} & \text{for loading tides} \\ -\frac{1}{8\pi G}\int_{\mathcal{V}} \nabla\varphi^*\cdot\nabla\varphi^{\mathrm{S}}\,\mathrm{d}^3\boldsymbol{r} + \frac{1}{8\pi G}\int_{\partial_+\mathcal{V}} \varphi^{\mathrm{S}}\boldsymbol{e}_r\cdot\nabla\varphi^*\mathrm{d}^2\boldsymbol{r} & \text{for body tides} \end{cases} . \end{aligned} \tag{37}$$

The elastic deformation energy is

$$\begin{aligned}\mathcal{E} \;&=\; -\frac{1}{2}\int_{\mathcal{V}} \boldsymbol{u}^* \cdot (\nabla\cdot \boldsymbol{\sigma})\,\mathrm{d}^3\boldsymbol{r} + \frac{1}{2}\int_{\partial\mathcal{V}} \boldsymbol{u}^* \cdot (\boldsymbol{e}_r \cdot \boldsymbol{\sigma})\mathrm{d}^2\boldsymbol{r} \\ &=\; \frac{1}{2}\int_{\mathcal{V}} \left\{ \lambda|\nabla\cdot \boldsymbol{u}|^2 + \frac{\mu}{2}\sum_{i,j=1}^{3} |\partial_i u_j + \partial_j u_i|^2 \right\} \mathrm{d}^3\boldsymbol{r}\end{aligned} \tag{38}$$

and the kinetic energy is

$$\mathcal{K} = \frac{1}{2}\int_{\mathcal{V}} \rho\dot{\boldsymbol{u}}^* \cdot \dot{\boldsymbol{u}}\,\mathrm{d}^3\boldsymbol{r}\;. \tag{39}$$

Finally, the potential energy of the load lift is

$$\mathcal{L} = g(a)\int_{\partial\mathcal{V}} (\boldsymbol{u}\cdot\boldsymbol{e}_r)\tau^*\mathrm{d}^2\boldsymbol{r} \tag{40}$$

while the kinetic energy of the load is small of third order.

For spherical degree n with the combined ansatz (11) and (17) the integral (34) is, executing the ϑ- and χ-integration and using Poisson's equation $g'(r) = 4\pi G\rho(r) - 2g(r)/r$, evaluated to

$$\begin{aligned}\mathcal{P}_n = \frac{1}{2}\int_0^a \Big\{ &(4\pi G\rho(r)r - 4g(r))\,|H_n(r)|^2 \\ &+n(n+1)g(r)\,[H_n(r)^*T_n(r) + H_n(r)T_n(r)^*]\Big\}\rho(r)r\mathrm{d}r\;,\end{aligned} \tag{41}$$

whereas, with the ansatz

$$\varphi^{\mathrm{P}} = C_{\mathrm{P}}\left(\frac{r}{a}\right)^n Y_{nm}(\vartheta,\chi)\exp(i\omega t)\;. \tag{42}$$

for body tides and

$$\varphi^{\mathrm{P}} = \begin{cases} \dfrac{4\pi Ga}{2n+1}\tau_{nm}\left(\dfrac{r}{a}\right)^n Y_{nm}(\vartheta,\chi)\exp(i\omega t) & \text{for} \quad r \le a \\ \dfrac{4\pi Ga}{2n+1}\tau_{nm}\left(\dfrac{a}{r}\right)^{n+1} Y_{nm}(\vartheta,\chi)\exp(i\omega t) & \text{for} \quad r > a \end{cases} \tag{43}$$

for loading tides, the energy $\mathcal{G}$ is evaluated from (35) to

$$\begin{aligned}\mathcal{G}_n \;&=\; -\frac{1}{2}\int_0^a \{rH_n(r)^*R_n'(r) + n(n+1)T_n(r)^*R_n(r)\}\,\rho(r)r\mathrm{d}r - \frac{1}{2}a^2R_n(a)\tau_{nm}^* \\ &=\; -\frac{1}{8\pi G}\Big\{\int_0^a \left[r^2|R_n'(r)|^2 + n(n+1)|R_n(r)|^2\right]\mathrm{d}r \\ &\qquad +aR_n(a)[(n+1)R_n(a) - (2n+1)C_{\mathrm{P}}]^*\Big\}\end{aligned} \tag{44}$$

where, for loading tides, $C_{\mathrm{P}} = 0$ is meant. The primary and secondary part of the this energy are

$$\mathcal{G}_n^{\mathrm{P}} = \begin{cases} -\dfrac{nC_{\mathrm{P}}}{2}\displaystyle\int_0^a \{H_n(r)^* + (n+1)T_n(r)^*\}\left(\frac{r}{a}\right)^n \rho(r)r\mathrm{d}r \\ \quad = -(2n+1)\dfrac{aC_{\mathrm{P}}}{8\pi G}\left[R_n(a) - C_{\mathrm{P}}\right]^* & \text{for body tides} \\ -\dfrac{2\pi Gan\tau_{nm}}{2n+1}\displaystyle\int_0^a \{H_n(r)^* + (n+1)T_n(r)^*\}\left(\frac{r}{a}\right)^n \rho(r)r\mathrm{d}r - \frac{a^2}{2}R_n(a)\tau_{nm}^* \\ \quad = -\dfrac{a^2}{2}\left[\tau_{nm}R_n(a)^* + \tau_{nm}^*R_n(a) - \dfrac{4\pi Ga}{2n+1}|\tau_{nm}|^2\right] & \text{for loading tides} \end{cases} \tag{45}$$

and

$$\mathcal{G}_n^{\mathrm{S}} = \begin{cases} -\dfrac{1}{2}\displaystyle\int_0^a \Big\{ H_n(r)^* \Big[rR_n'(r) - nC_{\mathrm{P}}\left(\dfrac{r}{a}\right)^n \Big] \\ \qquad +n(n+1)T_n(r)^* \Big[R_n(r) - C_{\mathrm{P}}\left(\dfrac{r}{a}\right)^n \Big] \Big\} \rho(r) r \mathrm{d}r \\ \quad = -\dfrac{1}{8\pi G} \Big\{ \displaystyle\int_0^a \Big\{ rR_n'(r)^* \Big[rR_n'(r) - nC_{\mathrm{P}}\left(\dfrac{r}{a}\right)^n \Big] \\ \qquad\qquad +n(n+1)R_n(r)^* \Big[R_n(r) - C_{\mathrm{P}}\left(\dfrac{r}{a}\right)^n \Big] \Big\} \mathrm{d}r \\ \qquad +a[(n+1)R_n(a) - (2n+1)C_{\mathrm{P}}]^* [R_n(a) - C_{\mathrm{P}}] \Big\} \quad \text{for body tides} \\ \\ -\dfrac{1}{2}\displaystyle\int_0^a \Big\{ H_n(r)^* \Big[rR_n'(r) - \dfrac{4\pi G a n}{2n+1}\tau_{nm}\left(\dfrac{r}{a}\right)^n \Big] \\ \qquad +n(n+1)T_n(r)^* \Big[R_n(r) - \dfrac{4\pi G a}{2n+1}\tau_{nm}\left(\dfrac{r}{a}\right)^n \Big] \Big\} \rho(r) r \mathrm{d}r \\ \quad = -\dfrac{1}{8\pi G} \Big\{ \displaystyle\int_0^a \Big\{ rR_n'(r)^* \Big[rR_n'(r) - \dfrac{4\pi G a n}{2n+1}\tau_{nm}\left(\dfrac{r}{a}\right)^n \Big] \\ \qquad\qquad +n(n+1)R_n(r)^* \Big[R_n(r) - \dfrac{4\pi G a}{2n+1}\tau_{nm}\left(\dfrac{r}{a}\right)^n \Big] \Big\} \mathrm{d}r \\ \qquad +\dfrac{a}{2n+1}\left[(2n+1)R_n(a) - 4\pi G a \tau_{nm}\right] \\ \qquad \times \left[(n+1)R_n(a) - 4\pi G a \tau_{nm}\right]^* \Big\} \quad \text{for loading tides} \end{cases} \tag{46}$$

respectively. From the ansatz (11) and (17) with (13) and (18) and the boundary condition (0.8a) the elastic deformation energy is evaluated as

$$\begin{aligned} \mathcal{E}_n = \frac{1}{2}\int_0^a \Big\{ & \lambda(r) \left| rH_n'(r) + 2H_n(r) - n(n+1)T_n(r) \right|^2 \\ & +\mu(r)\Big[2r^2|H_n'(r)|^2 + 4|H_n(r)|^2 - 2n(n+1)\left(H_n(r)^*T_n(r) + H_n(r)T_n(r)^*\right) \\ & +n(n+1)\left| rT_n'(r) - T_n(r) + H_n(r) \right|^2 + 2n(n+1)(n^2+n-1)|T_n(r)|^2 \\ & +n(n+1)\left| rW_n'(r) - W_n(r) \right|^2 + n(n+1)(n+2)(n-1)|W_n(r)|^2 \Big] \Big\} \mathrm{d}r \,, \end{aligned} \tag{47}$$

where the surface integrals η_{nm}, θ_{nm} and ζ_{nm} over the external force per area acting on the surface were introduced in order to express the boundary term and are defined below (p. 9, equation (0.71a), (0.71b) and (0.71c)). The kinetic energy obtains the value

$$\mathcal{K}_n = \frac{|\omega|^2}{2}\int_0^a \Big\{ |H_n(r)|^2 + n(n+1)\Big[|T_n(r)|^2 + |W_n(r)|^2 \Big] \Big\} \rho(r) r^2 \mathrm{d}r \tag{48}$$

and at last, the load-lift energy is

$$\mathcal{L}_n \;=\; a^2 g(a) H_n(a) \tau_{nm}^* \,. \tag{49}$$

The total inner potential energy of the mass redistribution within the earth then is

$$\left\{ \mathcal{P} + \mathcal{G}^{\mathrm{S}} + \mathcal{E} \right\} \cos^2(\omega(t - t_\circ)) \tag{50}$$

for a real time periodic deformation with circle frequency ω and phase $t_\circ$.

0.1.4 A special property of the differential equation system

The differential equation systems (15), (20), (23) and (31) have an important property related to the formal self adjointness of the differential equation (10) which has been first pointed out by Molodenskiy **?** for a slightly transformed equation system equivalent to (20). In the present formulation this property manifest itself by the similarity of the matrices in question with their negative transposed matrices:

$$-{\mathbf{A}_{\mathrm{T}}}^T \;=\; \mathbf{C}_{\mathrm{T}}\mathbf{A}_{\mathrm{T}}{\mathbf{C}_{\mathrm{T}}}^{-1} \,, \tag{0.51a}$$

$$-{\mathbf{A}_{\mathrm{S}}}^T \;=\; \mathbf{C}_{\mathrm{S}}\mathbf{A}_{\mathrm{S}}{\mathbf{C}_{\mathrm{S}}}^{-1} \,, \tag{0.51b}$$

$$-{\mathbf{A}_{\mathrm{L}}}^T \;=\; \mathbf{C}_{\mathrm{L}}\mathbf{A}_{\mathrm{L}}{\mathbf{C}_{\mathrm{L}}}^{-1} \,. \tag{0.51c}$$

The similarity transform is explicitly given by the matrices

$$\mathbf{C}_\mathrm{T} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ , \tag{0.52a}$$

$$\mathbf{C}_\mathrm{S} = \begin{pmatrix} 0 & -4\pi G n(n+1) & 0 & 0 & 0 & 0 \\ 4\pi G n(n+1) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4\pi G n(n+1) & 0 & 0 \\ 0 & 0 & 4\pi G n(n+1) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -n \\ 0 & 0 & 0 & 0 & n & 0 \end{pmatrix} \ , \tag{0.52b}$$

$$\mathbf{C}_\mathrm{L} = \begin{pmatrix} 0 & \omega^2 & 1 & 0 \\ -\omega^2 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \ . \tag{0.52c}$$

For $\mathbf{A}_0$ there does not exist such an equation with a constant non-singular matrix $\mathbf{C}_0$, but we have

$$-{\mathbf{A}_0}^T \mathbf{C}_0 = \mathbf{C}_0 \mathbf{A}_0 \tag{53}$$

with a matrix

$$\mathbf{C}_0 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ . \tag{54}$$

Of course, these matrices are only determined up to a constant matrix factor commuting with the matrix $\mathbf{A}_\mathrm{T}$, $\mathbf{A}_\mathrm{S}$, $\mathbf{A}_\mathrm{L}$, $\mathbf{A}_0$ respectively. These factors have been choosen such that the equality

$$\mathbf{x}_\mathrm{S}{}^T \mathbf{C}_\mathrm{S} \mathbf{y}_\mathrm{S} = \mathbf{x}_\mathrm{L}{}^T \mathbf{C}_\mathrm{L} \mathbf{y}_\mathrm{L} \tag{55}$$

holds by (33) at the boundary between a liquid and a solid layer.

By differentiating such scalar products and applying (15), (20), (31) it may be concluded from (0.51a) to (0.51c) that these scalar products must be constant. Due to the continuity (55) across the boundary, the constant must be the same in all layers. Taking only solutions **x** and **y** which go to zero at the center of the earth, we therefore have

$$\mathbf{x}_\mathrm{T}{}^T \mathbf{C}_\mathrm{T} \mathbf{y}_\mathrm{T} = 0 \ , \quad \mathbf{x}_\mathrm{S}{}^T \mathbf{C}_\mathrm{S} \mathbf{y}_\mathrm{S} = \mathbf{x}_\mathrm{L}{}^T \mathbf{C}_\mathrm{L} \mathbf{y}_\mathrm{L} = 0 \ . \tag{56}$$

From these equations there may be derived linear relations between Love and Shida numbers (see below, section 0.2.2).

0.1.5 Solutions for a homogeneous sphere

Solutions of the differential equation systems considered in section 0.1.2 in general diverge at the center of the earth. Therefore, these systems are numerically untractable there. To overcome this difficulty, a small sphere at the center of the earth may be taken to be homogeneous. For this homogeneous sphere the finite solutions of the differential equation systems shall be derived analytically. For a homogeneous sphere we have constant values of ρ, λ and μ and

$$g = \frac{4\pi}{3} G \rho r \ . \tag{57}$$

The toroidal solutions may be given directly in terms of spherical Bessel functions j_n of the first kind (cf. **?**, p. 437):

$$\mathbf{y}_\mathrm{T}(r) = C_\mathrm{T} \sqrt{r} \begin{pmatrix} j_n\left(\sqrt{\frac{\rho}{\mu}}\omega r\right) \\ \frac{\mu}{n+1}\left\{\sqrt{\frac{\rho}{\mu}}\omega r j_n'\left(\sqrt{\frac{\rho}{\mu}}\omega r\right) - j_n\left(\sqrt{\frac{\rho}{\mu}}\omega r\right)\right\} \end{pmatrix} \tag{58}$$

or

$$W_n(r) = C_\mathrm{T} j_n\left(\sqrt{\frac{\rho}{\mu}}\omega r\right) \ . \tag{59}$$

However, if the core-mantle boundary is supposed to be frictionless — and no other assumption is consistent with vanishing shear stress in the liquid core — the numerical integration has to be started at this boundary with the initial value

$$\mathbf{y}_\mathrm{T}{}^\circ = \begin{pmatrix} C_\mathrm{T}^\circ \\ 0 \end{pmatrix} . \tag{60}$$

In order to derive spheroidal solutions for $n > 0$ we decompose $\mathbf{A}_{\mathrm{S}}$ with g replaced by the expression (57) into

$$\mathbf{A}_{\mathrm{S}} = \mathbf{A}_{\mathrm{S}}^{(0)} + r\mathbf{A}_{\mathrm{S}}^{(1)} + r^2\mathbf{A}_{\mathrm{S}}^{(2)} \ , \tag{61}$$

where

$$\mathbf{A}_{\mathrm{S}}^{(0)} = \begin{pmatrix} \frac{-3\lambda+2\mu}{2(\lambda+2\mu)} & \frac{n+1}{\lambda+2\mu} & \frac{(n+1)\lambda}{\lambda+2\mu} & 0 & 0 & 0 \\ \frac{4\mu(3\lambda+2\mu)}{(n+1)(\lambda+2\mu)} & \frac{3\lambda-2\mu}{2(\lambda+2\mu)} & \frac{-2\mu(3\lambda+2\mu)}{\lambda+2\mu} & n & 0 & 0 \\ -n & 0 & \frac{3}{2} & \frac{n}{\mu} & 0 & 0 \\ \frac{-2\mu(3\lambda+2\mu)}{\lambda+2\mu} & \frac{-(n+1)\lambda}{\lambda+2\mu} & \frac{4\mu(n+1)(\lambda+\mu)}{\lambda+2\mu} - \frac{2\mu}{n} & -\frac{3}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\left(n+\frac{1}{2}\right) & 1 \\ 0 & 0 & 0 & 0 & 0 & n+\frac{1}{2} \end{pmatrix} , \tag{0.62a}$$

$$\mathbf{A}_{\mathrm{S}}^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \rho & -\frac{\rho}{n+1} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\rho & 0 \\ 4\pi G\rho & 0 & 0 & 0 & 0 & 0 \\ 4\pi G\rho(n+1) & 0 & -4\pi G\rho(n+1) & 0 & 0 & 0 \end{pmatrix} , \tag{0.62b}$$

$$\mathbf{A}_{\mathrm{S}}^{(2)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{16\pi G\rho^2+3\rho\omega^2}{3(n+1)} & 0 & \frac{4\pi}{3}G\rho^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{4\pi}{3}G\rho^2 & 0 & -\frac{\rho\omega^2}{n} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{0.62c}$$

are constant with respect to r. Then six linear independent solutions of (20) may be given as series

$$\mathbf{y_{\mathrm{S}}}^{(i)}(r) = r^{\alpha_i}\sum_{k=0}^{\infty} r^k \mathbf{b}_i^{(k)} \ , \tag{63}$$

where α_i are the eigenvalues of $\mathbf{A}_{\mathrm{S}}^{(0)}$ — $\alpha_1 = n - 1/2$, $\alpha_2 = n + 1/2$, $\alpha_3 = n + 3/2$, $\alpha_4 = -(n - 1/2)$, $\alpha_5 = -(n + 1/2)$, $\alpha_6 = -(n + 3/2)$ — and the vectors $\mathbf{b}_i^{(k)}$ must satisfy

$$\{(\alpha_i + k)\mathbf{1} - \mathbf{A}_{\mathrm{S}}^{(0)}\}\mathbf{b}_i^{(k)} = \mathbf{A}_{\mathrm{S}}^{(1)}\mathbf{b}_i^{(k-1)} + \mathbf{A}_{\mathrm{S}}^{(2)}\mathbf{b}_i^{(k-2)} \tag{64}$$

with $\mathbf{b}_i^{(j)} = \mathbf{0}$ for $j < 0$. Only the first three solutions are finite at the center of the earth, their coefficients explicitly read

$$\mathbf{b}_1^{(0)} = \begin{pmatrix} n \\ \frac{2n(n-1)}{n+1}\mu \\ n \\ 2(n-1)\mu \\ 0 \\ 0 \end{pmatrix} , \quad \mathbf{b}_1^{(1)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{4\pi}{3}G\rho n - \omega^2 \\ \frac{8\pi}{3}G\rho n(n-1) - (2n+1)\omega^2 \end{pmatrix} , \quad \mathbf{b}_1^{(k)} = \mathbf{0} \ \text{ for } \ k \geq 2 \tag{0.65a}$$

for $\alpha_1 = n - 1/2$ and

$$\mathbf{b}_2^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 2n+1 \end{pmatrix} , \quad \mathbf{b}_2^{(1)} = \frac{n\rho}{(n+1)[n(\lambda+\mu)-2\mu]}\begin{pmatrix} 0 \\ -\lambda \\ 1 \\ \mu \\ 0 \\ 0 \end{pmatrix} ,$$

$$\mathbf{b}_2^{(k)} = \left[\left(n+k+\tfrac{1}{2}\right)\mathbf{1} - \mathbf{A}_{\mathrm{S}}^{(0)}\right]^{-1}\left\{\mathbf{A}_{\mathrm{S}}^{(1)}\mathbf{b}_2^{(k-1)} + \mathbf{A}_{\mathrm{S}}^{(2)}\mathbf{b}_2^{(k-2)}\right\} \quad \text{for } \ k \geq 2 \tag{0.65b}$$

for $\alpha_2 = n + 1/2$ and

$$\mathbf{b}_3^{(0)} = \begin{pmatrix} (n+1)[n(\lambda+\mu) - 2\mu] \\ 2\mu[n(n-1)(\lambda+\mu) - (3\lambda+2\mu)] \\ n[(n+1)(\lambda+\mu) + 2(\lambda+2\mu)] \\ 2\mu[(n+1)^2(\lambda+\mu) - (\lambda+2\mu)] \\ 0 \\ 0 \end{pmatrix} ,$$

$$\mathbf{b}_3^{(1)} = -4\pi G\rho(n+1) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \mu \\ [n(\lambda+3\mu)+\mu] \end{pmatrix} , \tag{0.65c}$$

$$\mathbf{b}_3^{(k)} = \left[\left(n+k+\tfrac{3}{2}\right)\mathbf{1} - \mathbf{A}_\mathrm{S}^{(0)}\right]^{-1} \left\{\mathbf{A}_\mathrm{S}^{(1)}\mathbf{b}_3^{(k-1)} + \mathbf{A}_\mathrm{S}^{(2)}\mathbf{b}_3^{(k-2)}\right\} \quad \text{for } k \geq 2$$

for $\alpha_3 = n + 3/2$.

In the case $n = 0$ the finite solution may again be given explicitely in terms of the spherical Bessel functions j_1 and j_0 of the first kind:

$$\mathbf{y}_0 = C_\mathrm{S} \begin{pmatrix} \sqrt{r}j_1(\kappa r) \\ \sqrt{r}\left\{(\lambda+2\mu)\kappa r j_1'(\kappa r) - 2\lambda j_1(\kappa r)\right\} \\ \frac{4\pi G\rho}{\kappa}[1 - j_0(\kappa r)] \end{pmatrix} \tag{66}$$

or

$$\begin{aligned} H_0(r) &= C_\mathrm{S} j_1(\kappa r) \\ R_0(r) &= \tfrac{4\pi G\rho}{\kappa} C_\mathrm{S}[1 - j_0(\kappa r)] \end{aligned} \tag{67}$$

with

$$\kappa = \sqrt{\frac{16\pi G\rho^2 + 3\omega^2\rho}{3(\lambda+2\mu)}} . \tag{68}$$

In the case of a liquid the Adams-Williamson condition (25) is only consistent with homogeneity if the liquid is incompressible, i.e. if $\lambda = \infty$. Under this circumstances there are two linear independent finite solutions, which can be expressed by a power of r:

$$\mathbf{y}_\mathrm{L}{}^{(1)}(r) = r^{n+1/2} \begin{pmatrix} 4\pi G\rho \\ 1 \\ 0 \\ -4\pi G\rho \end{pmatrix} , \quad \mathbf{y}_\mathrm{L}{}^{(2)}(r) = r^{n+1/2} \begin{pmatrix} 0 \\ 0 \\ n \\ 2n+1 \end{pmatrix} . \tag{69}$$

The general finite solution then reads

$$\begin{aligned} H_n(r) &= C_\mathrm{L}^{(1)} n r^{n-1} , \\ T_n(r) &= C_\mathrm{L}^{(1)} r^{n-1} , \\ R_n(r) &= C_\mathrm{L}^{(2)} r^n . \end{aligned} \tag{0.70a}$$

0.2 Body tides and surface load

0.2.1 Spherical harmonic expansion of surface loads

As the first order differential equation system has been decomposed into toroidal and speroidal harmonic parts by (11) and (17) the external surface force $\boldsymbol{F}$ and the surface mass density τ have to be expanded likewise. Since the spherical harmonics (12) constitute an orthonormal system, this is easily accomplished by surface integration. For a periodic load $\boldsymbol{F} = \boldsymbol{F}^\circ \exp(i\omega t)$, $\tau = \tau^\circ \exp(i\omega t)$ with circle frequency ω we set

$$\eta_{nm} \stackrel{\text{def}}{=} \int_0^\pi \int_0^{2\pi} F_r^\circ(\vartheta,\chi) Y_{nm}^*(\vartheta,\chi) \sin(\vartheta)\, \mathrm{d}\chi \mathrm{d}\vartheta , \tag{0.71a}$$

$$\theta_{nm} \stackrel{\text{def}}{=} \frac{-1}{n(n+1)} \int_0^\pi \int_0^{2\pi} \left\{\frac{\partial}{\partial\vartheta}[\sin(\vartheta) F_\vartheta^\circ(\vartheta,\chi)] + \frac{\partial}{\partial\chi} F_\chi^\circ(\vartheta,\chi)\right\} Y_{nm}^*(\vartheta,\chi)\, \mathrm{d}\chi \mathrm{d}\vartheta , \tag{0.71b}$$

$$\zeta_{nm} \stackrel{\text{def}}{=} \frac{1}{n(n+1)} \int_0^\pi \int_0^{2\pi} \left\{\frac{\partial}{\partial\chi} F_\vartheta^\circ(\vartheta,\chi) - \frac{\partial}{\partial\vartheta}[\sin(\vartheta) F_\chi^\circ(\vartheta,\chi)]\right\} Y_{nm}^*(\vartheta,\chi)\, \mathrm{d}\chi \mathrm{d}\vartheta \tag{0.71c}$$

and

$$\tau_{nm} \stackrel{\text{def}}{=} \int_0^{\pi}\int_0^{2\pi} \tau^\circ(\vartheta,\chi) Y^*_{nm}(\vartheta,\chi)\sin(\vartheta)\,\mathrm{d}\chi\mathrm{d}\vartheta \ . \tag{0.71d}$$

The definitions (0.71b) and (0.71c) amount to decompose the horizontal component of the force $\boldsymbol{F}$ into spheroidal harmonic parts θ_{nm} and toroidal harmonic parts ζ_{nm}.

For the combined ansatz (11) and (17) the incremental pseudo-stress tensor components are related to the components of the solution vectors (14) and (19) as follows:

$$\begin{aligned} \sigma_{rr} &= \tfrac{n+1}{r^{3/2}}\, Y_{nm}(\vartheta,\chi)\exp(i\omega t)\,\mathrm{y}_{\mathrm{S},2} \ , \\ \tfrac{-1}{\sin(\vartheta)}\left\{\frac{\partial}{\partial\vartheta}[\sin(\vartheta)\sigma_{r\vartheta}] + \frac{\partial}{\partial\chi}\sigma_{r\chi}\right\} &= \tfrac{n(n+1)}{r^{3/2}}\, Y_{nm}(\vartheta,\chi)\exp(i\omega t)\,\mathrm{y}_{\mathrm{S},4} \ , \\ \tfrac{1}{\sin(\vartheta)}\left\{\frac{\partial}{\partial\chi}\sigma_{r\vartheta} - \frac{\partial}{\partial\vartheta}[\sin(\vartheta)\sigma_{r\chi}]\right\} &= \tfrac{n(n+1)^2}{r^{3/2}}\, Y_{nm}(\vartheta,\chi)\exp(i\omega t)\,\mathrm{y}_{\mathrm{T},2} \ . \end{aligned} \tag{72}$$

From these relations we conclude that the surface boundary condition (0.8a) becomes

$$\mathrm{y}_{\mathrm{S},2}\big|_{r=a-0} = \frac{a^{3/2}}{n+1}\eta_{nm} \ , \tag{0.73a}$$

$$\mathrm{y}_{\mathrm{S},4}\big|_{r=a-0} = a^{3/2}\,\theta_{nm} \ , \tag{0.73b}$$

$$\mathrm{y}_{\mathrm{T},2}\big|_{r=a-0} = \frac{a^{3/2}}{n+1}\zeta_{nm} \ . \tag{0.73c}$$

In order to evaluate the surface boundary condition (0.8b), we remark that φ satisfying the Laplace equation outside the earth must have the form $\varphi \propto r^{-(n+1)}Y_{nm}(\vartheta,\chi)$ if it is to vanish at infinity, i.e. no primary potential from sources outside the earth is present, and that φ is continuous through the surface. Therefore $[\partial\varphi/\partial r]_+ = -(n+1)[\varphi/r]_-$ and equation (0.8b) may be transformed to

$$\mathrm{y}_{\mathrm{S},6}\big|_{r=a-0} = 4\pi G a^{3/2}\tau_{nm} \ . \tag{0.73d}$$

If the vertical component of the external surface force is solely due to the material on the surface which must have the same vertical velocity $\dot{u}_r$ as the surface themselves, the relation

$$\eta_{nm} = -g(a)\tau_{nm} \tag{74}$$

must be satisfied. This is so because the vertical acceleration $\ddot{u}_r$ times the surface mass density τ is small of second order and must be neglected in the linearized theory studied here.

0.2.2 Love-Shida numbers for body tides and surface load

The Love numbers h_n, k_n and the Shida number ℓ_n are defined to be the appropriately normalized dimensionless answer of the system to a primary excitation potential of equation (42) **?**. The incremental potential then is $\varphi = \varphi^{\mathrm{P}} + \varphi^{\mathrm{S}}$ with $\varphi^{\mathrm{S}} = C_{\mathrm{S}}\left(\frac{a}{r}\right)^{n+1} Y_{nm}(\vartheta,\chi)\exp(i\omega t)$ outside the earth. We then have $R_n(a) = C_{\mathrm{P}} + C_{\mathrm{S}}$ and, according to the boundary condition (0.8c), $aR'_n(a) - 4\pi G\rho(a)aH_n(a) = nC_{\mathrm{P}} - (n+1)C_{\mathrm{S}}$. Therefore, $\mathbf{y}_{\mathrm{S}6}(a) = (2n+1)\sqrt{a}\,C_{\mathrm{P}}$. Now the Love-Shida numbers are defined as

$$h_n(r) = \frac{g(a)H_n(r)}{C_{\mathrm{P}}} = (2n+1)g(a)\sqrt{\frac{a}{r}}\frac{\mathbf{y}_{\mathrm{S}1}(r)}{\mathbf{y}_{\mathrm{S}6}(a)} \ , \tag{0.75a}$$

$$\ell_n(r) = \frac{g(a)T_n(r)}{C_{\mathrm{P}}} = \frac{2n+1}{n}g(a)\sqrt{\frac{a}{r}}\frac{\mathbf{y}_{\mathrm{S}3}(r)}{\mathbf{y}_{\mathrm{S}6}(a)} \ , \tag{0.75b}$$

$$k_n(r) = \frac{R_n(r)}{C_{\mathrm{P}}} - \left(\frac{r}{a}\right)^n = (2n+1)\sqrt{\frac{a}{r}}\frac{\mathbf{y}_{\mathrm{S}5}(r)}{\mathbf{y}_{\mathrm{S}6}(a)} - \left(\frac{r}{a}\right)^n \ , \tag{0.75c}$$

for a solution $\mathbf{y}_{\mathrm{S}}$ with $\mathbf{y}_{\mathrm{S}2}(a) = \mathbf{y}_{\mathrm{S}4}(a) = 0$ because of the boundary condition (0.8a) with $\boldsymbol{F} = 0$. At the surface these numbers have the values

$$h_n = \frac{g(a)H_n(a)}{C_{\mathrm{P}}} = (2n+1)g(a)\frac{\mathbf{y}_{\mathrm{S}1}(a)}{\mathbf{y}_{\mathrm{S}6}(a)} \ , \tag{0.76a}$$

$$\ell_n = \frac{g(a)T_n(a)}{C_{\mathrm{P}}} = \frac{2n+1}{n}g(a)\frac{\mathbf{y}_{\mathrm{S}3}(a)}{\mathbf{y}_{\mathrm{S}6}(a)} \ , \tag{0.76b}$$

$$k_n = \frac{R_n(a)}{C_{\mathrm{P}}} - 1 = (2n+1)\frac{\mathbf{y}_{\mathrm{S}5}(a)}{\mathbf{y}_{\mathrm{S}6}(a)} - 1 \ . \tag{0.76c}$$

There are also defined three load factors **???** for a surface mass load with $\eta_{nm} = -g(a)\tau_{nm}$ and $\theta_{nm} = \zeta_{nm} = 0$:

$$h'_n(r) = \frac{(2n+1)g(a)}{4\pi Ga}\frac{H_n(r)}{\tau_{nm}} = (2n+1)g(a)\sqrt{\frac{a}{r}}\frac{\mathbf{y}_{\mathrm{S}1}(r)}{\mathbf{y}_{\mathrm{S}6}(a)} , \tag{0.77a}$$

$$\ell'_n(r) = \frac{(2n+1)g(a)}{4\pi Ga}\frac{T_n(r)}{\tau_{nm}} = \frac{2n+1}{n}g(a)\sqrt{\frac{a}{r}}\frac{\mathbf{y}_{\mathrm{S}3}(r)}{\mathbf{y}_{\mathrm{S}6}(a)} , \tag{0.77b}$$

$$k'_n(r) = \frac{2n+1}{4\pi Ga}\frac{R_n(r)}{\tau_{nm}} - \left(\frac{r}{a}\right)^n = (2n+1)\sqrt{\frac{a}{r}}\frac{\mathbf{y}_{\mathrm{S}5}(r)}{\mathbf{y}_{\mathrm{S}6}(a)} - \left(\frac{r}{a}\right)^n . \tag{0.77c}$$

Here, the solution $\mathbf{y}_\mathrm{S}$ must satisfy $\mathbf{y}_{\mathrm{S}2}(a) = -\frac{g(a)}{4\pi G(n+1)}\mathbf{y}_{\mathrm{S}6}(a)$ and $\mathbf{y}_{\mathrm{S}4}(a) = 0$ because of the boundary conditions (0.73a), (0.73d) and (0.73b). At the surface these factors become

$$h'_n = \frac{2n+1}{4\pi a^3}M\frac{H_n(a)}{\tau_{nm}} = (2n+1)g(a)\frac{\mathbf{y}_{\mathrm{S}1}(a)}{\mathbf{y}_{\mathrm{S}6}(a)} , \tag{0.78a}$$

$$\ell'_n = \frac{2n+1}{4\pi a^3}M\frac{T_n(a)}{\tau_{nm}} = \frac{2n+1}{n}g(a)\frac{\mathbf{y}_{\mathrm{S}3}(a)}{\mathbf{y}_{\mathrm{S}6}(a)} , \tag{0.78b}$$

$$k'_n = \frac{2n+1}{4\pi Ga}\frac{R_n(a)}{\tau_{nm}} - 1 = (2n+1)\frac{\mathbf{y}_{\mathrm{S}5}(a)}{\mathbf{y}_{\mathrm{S}6}(a)} - 1 , \tag{0.78c}$$

where M is the mass of the earth.

A unit point mass located at colatitude $\vartheta_\circ$ and longitude $\chi_\circ$ has sherical harmonic expansion

$$\delta(\vartheta,\chi|\vartheta_\circ,\chi_\circ) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} Y_{nm}(\vartheta,\chi)Y^*_{nm}(\vartheta_\circ,\chi_\circ) . \tag{79}$$

If the unit point mass is located at the north pole, which, because of spherical symmetry, might be achieved by taking the coordinate system appropriate, the double sum may be reduced to a single sum over Legendre polynomials

$$\delta(\vartheta,\chi|0) = \sum_{n=0}^{\infty}\frac{2n+1}{4\pi}P_n(\cos\vartheta) . \tag{80}$$

The system therefore reacts on a load $\tau(\vartheta,\chi) = \frac{M_\circ}{a^2}\delta(\vartheta,\chi|0)\exp(i\omega t)$, which gives $\tau_{nm} = \frac{M_\circ}{a^2}\sqrt{\frac{2n+1}{4\pi}}\delta_{m0}$, with elongations

$$u_r(a,\vartheta,\chi) = \frac{M_\circ a}{M}\sum_{n=0}^{\infty}h'_n P_n(\cos\vartheta)\exp(i\omega t) , \tag{0.81a}$$

$$u_\vartheta(a,\vartheta,\chi) = \frac{M_\circ a}{M}\sum_{n=0}^{\infty}\ell'_n\frac{\partial}{\partial\vartheta}P_n(\cos\vartheta)\exp(i\omega t) , \tag{0.81b}$$

$u_\chi(a,\vartheta,\chi) = 0$ and incremental potential at the undisturbed surface

$$\varphi(a,\vartheta,\chi) = \frac{M_\circ G}{a}\sum_{n=0}^{\infty}(k'_n+1)P_n(\cos\vartheta)\exp(i\omega t) . \tag{82}$$

For tangential forces there might be defined four more dimensionless coefficients, three for spheroidal forcing **??**:

$$h''_n(r) = \frac{2n+1}{n(n+1)}\frac{g(a)^2}{4\pi Ga}\frac{H_n(r)}{\theta_{nm}} = \frac{2n+1}{n(n+1)}\frac{g(a)^2}{4\pi G}\sqrt{\frac{a}{r}}\frac{\mathbf{y}_{\mathrm{S}1}(r)}{\mathbf{y}_{\mathrm{S}4}(a)} , \tag{0.83a}$$

$$\ell''_n(r) = \frac{(2n+1)}{n(n+1)}\frac{g(a)^2}{4\pi Ga}\frac{T_n(r)}{\theta_{nm}} = \frac{2n+1}{n^2(n+1)}\frac{g(a)^2}{4\pi G}\sqrt{\frac{a}{r}}\frac{\mathbf{y}_{\mathrm{S}3}(r)}{\mathbf{y}_{\mathrm{S}4}(a)} , \tag{0.83b}$$

$$k''_n(r) = \frac{2n+1}{n(n+1)}\frac{g(a)}{4\pi Ga}\frac{R_n(r)}{\theta_{nm}} = \frac{2n+1}{n(n+1)}\frac{g(a)}{4\pi G}\sqrt{\frac{a}{r}}\frac{\mathbf{y}_{\mathrm{S}5}(r)}{\mathbf{y}_{\mathrm{S}4}(a)} , \tag{0.83c}$$

for a solution $\mathbf{y}_\mathrm{S}$ with $\mathbf{y}_{\mathrm{S}2}(a) = \mathbf{y}_{\mathrm{S}6}(a) = 0$ because of (0.73a) and (0.73d). At the surface these coefficients read

$$h''_n = \frac{2n+1}{n(n+1)}\frac{GM^2}{4\pi a^5}\frac{H_n(a)}{\theta_{nm}} = \frac{2n+1}{n(n+1)}\frac{Mg(a)}{4\pi a^2}\frac{\mathbf{y}_{\mathrm{S}1}(a)}{\mathbf{y}_{\mathrm{S}4}(a)} , \tag{0.84a}$$

$$\ell''_n = \frac{2n+1}{n(n+1)}\frac{GM^2}{4\pi a^5}\frac{T_n(a)}{\theta_{nm}} = \frac{2n+1}{n^2(n+1)}\frac{Mg(a)}{4\pi a^2}\frac{\mathbf{y}_{\mathrm{S}3}(a)}{\mathbf{y}_{\mathrm{S}4}(a)} , \tag{0.84b}$$

$$k''_n = \frac{2n+1}{n(n+1)}\frac{M}{4\pi a^3}\frac{R_n(a)}{\theta_{nm}} = \frac{2n+1}{n(n+1)}\frac{M}{4\pi a^2}\frac{\mathbf{y}_{\mathrm{S}5}(a)}{\mathbf{y}_{\mathrm{S}4}(a)} . \tag{0.84c}$$

Finally, there is one coefficient for toroidal forcing **?**:

$$f_n(r) = \frac{g(a)^2}{4\pi G a}\,\frac{W_n(r)}{\zeta_{nm}} \;=\; \frac{g(a)^2}{4\pi G(n+1)}\sqrt{\frac{a}{r}}\,\frac{\mathbf{y}_{\mathrm{T}1}(r)}{\mathbf{y}_{\mathrm{T}2}(a)}\;, \tag{85}$$

which at the surface has the value

$$f_n = \frac{GM^2}{4\pi a^5}\,\frac{W_n(a)}{\zeta_{nm}} \;=\; \frac{Mg(a)}{4\pi(n+1)a^2}\,\frac{\mathbf{y}_{\mathrm{T}1}(a)}{\mathbf{y}_{\mathrm{T}2}(a)}\;. \tag{86}$$

The nine coefficients for spheroidal forcing are not independent. Applying equation (56) to any pair of the three solutions

$$\mathbf{y} = \begin{pmatrix} h_n \\ 0 \\ n\ell_n \\ 0 \\ (1+k_n)g \\ (2n+1)g \end{pmatrix}\;,\quad \mathbf{y}' = \begin{pmatrix} h'_n \\ \frac{-(2n+1)Mg}{4\pi(n+1)a^2} \\ n\ell'_n \\ 0 \\ (1+k'_n)g \\ (2n+1)g \end{pmatrix}\;,\quad \mathbf{y}'' = \begin{pmatrix} h''_n \\ 0 \\ n\ell''_n \\ \frac{(2n+1)Mg}{4\pi n(n+1)a^2} \\ k''_n g \\ 0 \end{pmatrix}\;, \tag{87}$$

given at the surface of the earth, the three linear relations

$$k'_n \;=\; k_n - h_n \tag{0.88a}$$
$$k''_n \;=\; \ell_n \tag{0.88b}$$
$$h''_n \;=\; k''_n - \ell'_n \;=\; \ell_n - \ell'_n \tag{0.88c}$$

valid at the surface may be derived **?**.

Furthermore, evaluating equation (9), there may be derived the following integral relations for the Love-Shida numbers:

$$k_n(a) \;=\; \frac{4\pi G n}{(2n+1)g(a)}\int_0^a \{h_n(r) + (n+1)\ell_n(r)\}\left(\frac{r}{a}\right)^{n+1}\rho(r)\mathrm{d}r\;, \tag{0.89a}$$

$$k'_n(a) \;=\; \frac{4\pi G n}{(2n+1)g(a)}\int_0^a \{h'_n(r) + (n+1)\ell'_n(r)\}\left(\frac{r}{a}\right)^{n+1}\rho(r)\mathrm{d}r\;, \tag{0.89b}$$

$$k''_n(a) \;=\; \frac{4\pi G n}{(2n+1)g(a)}\int_0^a \{h''_n(r) + (n+1)\ell''_n(r)\}\left(\frac{r}{a}\right)^{n+1}\rho(r)\mathrm{d}r\;. \tag{0.89c}$$

0.2.3 Dimensionless representation of energy integrals

The energy integrals of section 0.1.3 may also be expressed entirely in terms of Love-Shida numbers. In order to do so, we choose the dimensionless normalizations

$$\overline{\mathcal{P}}_n \overset{\mathrm{def}}{=} \frac{4\pi G}{a|C_{\mathrm{P}}|^2}\,\mathcal{P}_n\;,\qquad \overline{\mathcal{G}}{}^{\mathrm{P}}_n \overset{\mathrm{def}}{=} \frac{4\pi G}{a|C_{\mathrm{P}}|^2}\,\mathcal{G}^{\mathrm{P}}_n\;,\qquad \overline{\mathcal{G}}{}^{\mathrm{S}}_n \overset{\mathrm{def}}{=} \frac{4\pi G}{a|C_{\mathrm{P}}|^2}\,\mathcal{G}^{\mathrm{S}}_n\;,$$
$$\overline{\mathcal{E}}_n \overset{\mathrm{def}}{=} \frac{4\pi G}{a|C_{\mathrm{P}}|^2}\,\mathcal{E}_n\;,\qquad \overline{\mathcal{K}}_n \overset{\mathrm{def}}{=} \frac{4\pi G}{a|C_{\mathrm{P}}|^2}\,\mathcal{K}_n\;, \tag{90}$$

for body tides,

$$\overline{\mathcal{P}}{}'_n \overset{\mathrm{def}}{=} \frac{2n+1}{4\pi G a^3|\tau_{nm}|^2}\,\mathcal{P}_n\;,\quad \overline{\mathcal{G}}{}^{\mathrm{P}\prime}_n \overset{\mathrm{def}}{=} \frac{2n+1}{4\pi G a^3|\tau_{nm}|^2}\,\mathcal{G}^{\mathrm{P}}_n\;,\quad \overline{\mathcal{G}}{}^{\mathrm{S}\prime}_n \overset{\mathrm{def}}{=} \frac{2n+1}{4\pi G a^3|\tau_{nm}|^2}\,\mathcal{G}^{\mathrm{S}}_n\;,$$
$$\overline{\mathcal{E}}{}'_n \overset{\mathrm{def}}{=} \frac{2n+1}{4\pi G a^3|\tau_{nm}|^2}\,\mathcal{E}_n\;,\quad \overline{\mathcal{K}}{}'_n \overset{\mathrm{def}}{=} \frac{2n+1}{4\pi G a^3|\tau_{nm}|^2}\,\mathcal{K}_n\;,\quad \overline{\mathcal{L}}{}'_n \overset{\mathrm{def}}{=} \frac{2n+1}{4\pi G a^3|\tau_{nm}|^2}\,\mathcal{L}_n \tag{91}$$

in the case of loading tides,

$$\overline{\mathcal{P}}{}''_n \overset{\mathrm{def}}{=} \frac{(2n+1)^2 g(a)^2}{4\pi G a^3 n^2(n+1)^2|\theta_{nm}|^2}\,\mathcal{P}_n\;,\qquad \overline{\mathcal{G}}{}''_n \overset{\mathrm{def}}{=} \frac{(2n+1)^2 g(a)^2}{4\pi G a^3 n^2(n+1)^2|\theta_{nm}|^2}\,\mathcal{G}_n\;,$$
$$\overline{\mathcal{E}}{}''_n \overset{\mathrm{def}}{=} \frac{(2n+1)^2 g(a)^2}{4\pi G a^3 n^2(n+1)^2|\theta_{nm}|^2}\,\mathcal{E}_n\;,\qquad \overline{\mathcal{K}}{}''_n \overset{\mathrm{def}}{=} \frac{(2n+1)^2 g(a)^2}{4\pi G a^3 n^2(n+1)^2|\theta_{nm}|^2}\,\mathcal{K}_n \tag{92}$$

for spheroidal surface forces and

$$\overline{\mathcal{E}}{}^{(\mathrm{t})}_n \overset{\mathrm{def}}{=} \frac{g(a)^2}{4\pi G a^3 n(n+1)\zeta_{nm}^2}\,\mathcal{E}_n\;,\qquad \overline{\mathcal{K}}{}^{(\mathrm{t})}_n \overset{\mathrm{def}}{=} \frac{g(a)^2}{4\pi G a^3 n(n+1)\zeta_{nm}^2}\,\mathcal{K}_n \tag{93}$$

in the case of toroidal surface forces. Then, with the definitions

$$\bar{\rho}(x) \stackrel{\text{def}}{=} \frac{4\pi G a}{g(a)}\rho(xa) \; , \quad \bar{g}(x) \stackrel{\text{def}}{=} \frac{g(xa)}{g(a)} \; , \tag{0.94a}$$

$$\bar{\lambda}(x) \stackrel{\text{def}}{=} \frac{4\pi G}{g(a)^2}\lambda(xa) \; , \quad \bar{\mu}(x) \stackrel{\text{def}}{=} \frac{4\pi G}{g(a)^2}\mu(xa) \; , \tag{0.94b}$$

the energy integrals (41) and (44) to (47) for body tides then, performing some partial integrations, read

$$\begin{aligned}\overline{\mathcal{P}}_n = \frac{1}{2}\int_0^1 \Big\{ & (x\bar{\rho}(x) - 4\bar{g}(x))|h_n(x)|^2 \\ & + n(n+1)\bar{g}(x)\left[h_n(x)^*\ell_n(x) + h_n(x)\ell_n(x)^*\right]\Big\}\bar{\rho}(x)x\mathrm{d}x \; ,\end{aligned} \tag{0.95a}$$

$$\begin{aligned}\overline{\mathcal{G}}_n^{\,\mathrm{P}} &= -\frac{1}{2}\int_0^1 n\left\{h_n(x)^* + (n+1)\ell_n(x)^*\right\}\bar{\rho}(x)x^{n+1}\mathrm{d}x \\ &= -\frac{2n+1}{2}k_n(1)^* \; ,\end{aligned} \tag{0.95b}$$

$$\begin{aligned}\overline{\mathcal{G}}_n^{\,\mathrm{S}} &= -\frac{1}{2}\int_0^1 \left\{ x h_n(x)^* \frac{\mathrm{d}k_n(x)}{\mathrm{d}x} + n(n+1)\ell_n(x)^* k_n(x)\right\}\bar{\rho}(x)x\mathrm{d}x \\ &= -\frac{1}{2}\left(\int_0^1 \left\{\left|x\frac{\mathrm{d}k_n(x)}{\mathrm{d}x}\right|^2 + n(n+1)|k_n(x)|^2\right\}\mathrm{d}x + (n+1)|k_n(1)|^2\right) \; ,\end{aligned} \tag{0.95c}$$

$$\begin{aligned}\overline{\mathcal{E}}_n = \frac{1}{2}\int_0^1 \Bigg\{ & \bar{\lambda}(x)\left|x\frac{\mathrm{d}h_n(x)}{\mathrm{d}x} + 2h_n(x) - n(n+1)\ell_n(x)\right|^2 + \bar{\mu}(x)\Bigg[2\left|x\frac{\mathrm{d}h_n(x)}{\mathrm{d}x}\right|^2 \\ & + 4|h_n(x)|^2 - 2n(n+1)\Big[h_n(x)^*\ell_n(x) + h_n(x)\ell_n(x)^*\Big] \\ & + n(n+1)\Bigg[\left|x\frac{\mathrm{d}\ell_n(x)}{\mathrm{d}x} - \ell_n(x) + h_n(x)\right|^2 \\ & \qquad + 2(n^2+n-1)|\ell_n(x)|^2\Bigg]\Bigg]\Bigg\}\mathrm{d}x \; ,\end{aligned} \tag{0.95d}$$

$$\overline{\mathcal{K}}_n = \frac{1}{2}\frac{a|\omega|^2}{g(a)}\int_0^1 \left\{|h_n(x)|^2 + n(n+1)|\ell_n(x)|^2\right\}\bar{\rho}(x)x^2\mathrm{d}x \; . \tag{0.95e}$$

For loading tides one accordingly has

$$\begin{aligned}\overline{\mathcal{P}}'_n = \frac{1}{2}\int_0^1 \Big\{ & (x\bar{\rho}(x) - 4\bar{g}(x))|h'_n(x)|^2 \\ & + n(n+1)\bar{g}(x)\left[h'_n(x)^*\ell'_n + h'_n(x)\ell'_n(x)^*\right]\Big\}\frac{\bar{\rho}(x)x}{2n+1}\mathrm{d}x \; ,\end{aligned} \tag{0.96a}$$

$$\begin{aligned}\overline{\mathcal{G}}_n^{\,\mathrm{P}'} &= -\frac{1}{2}\left(\int_0^1 \left\{h'_n(x)^* + (n+1)\ell'_n(x)^*\right\}\frac{n\bar{\rho}(x)x^{n+1}}{2n+1}\mathrm{d}x + [k'_n(1)+1]\right) \\ &= -\frac{1}{2}\left(k'_n(1) + k'_n(1)^* + 1\right) \; ,\end{aligned} \tag{0.96b}$$

$$\begin{aligned}\overline{\mathcal{G}}_n^{\,\mathrm{S}'} &= -\frac{1}{2}\int_0^1 \left\{ x h'_n(x)^* \frac{\mathrm{d}k'_n(x)}{\mathrm{d}x} + n(n+1)\ell'_n(x)^* k'_n(x)\right\}\frac{\bar{\rho}(x)x}{2n+1}\mathrm{d}x \\ &= -\frac{1}{2(2n+1)}\Bigg(\int_0^1 \left\{\left|x\frac{\mathrm{d}k'_n(x)}{\mathrm{d}x}\right|^2 + n(n+1)|k'_n(x)|^2\right\}\mathrm{d}x \\ &\qquad - (n+1)|k'_n(1)|^2\Bigg) \; ,\end{aligned} \tag{0.96c}$$

$$\overline{\mathcal{E}}'_n = \frac{1}{2(2n+1)} \int_0^1 \Bigg\{ \bar{\lambda}(x) \left| x\frac{\mathrm{d}h'_n(x)}{\mathrm{d}x} + 2h'_n(x) - n(n+1)\ell'_n(x) \right|^2 + \bar{\mu}(x) \Bigg[2 \left| x\frac{\mathrm{d}h'_n(x)}{\mathrm{d}x} \right|^2 + 4|h'_n(x)|^2 - 2n(n+1)\left[h'_n(x)^* \ell'_n(x) + h'_n(x)\ell'_n(x)^* \right] + n(n+1)\Bigg[\left| x\frac{\mathrm{d}\ell'_n(x)}{\mathrm{d}x} - \ell'_n(x) + h'_n(x) \right|^2 + 2(n^2+n-1)|\ell'_n(x)|^2 \Bigg]\Bigg] \Bigg\} \mathrm{d}x \;, \tag{0.96d}$$

$$\overline{\mathcal{K}}'_n = \frac{1}{2(2n+1)} \frac{a|\omega|^2}{g(a)} \int_0^1 \left\{ |h'_n(x)|^2 + n(n+1)|\ell'_n(x)|^2 \right\} \bar{\rho}(x) x^2 \mathrm{d}x \tag{0.96e}$$

and in this case there is an aditional load-lift energy (49) which may be given in the dimensionless form (91) as

$$\overline{\mathcal{L}}'_n \;=\; h'_n(1) \;. \tag{0.96f}$$

For spheroidal surface forces the dimensionless energy integrals may be evaluated as

$$\overline{\mathcal{P}}''_n = \frac{1}{2} \int_0^1 \Big\{ (x\bar{\rho}(x) - 4\bar{g}(x))|h''_n(x)|^2 + n(n+1)\bar{g}(x) \left[h''_n(x)^* \ell''_n + h''_n(x)\ell''_n(x)^* \right] \Big\} \bar{\rho}(x) x \mathrm{d}x \;, \tag{0.97a}$$

$$\overline{\mathcal{G}}''_n = -\frac{1}{2} \int_0^1 \left\{ x h''_n(x)^* \frac{\mathrm{d}k''_n(x)}{\mathrm{d}x} + n x^n + n(n+1)\ell''_n(x)^* k''_n(x) \right\} \bar{\rho}(x) x \mathrm{d}x$$
$$= -\frac{1}{2} \left(\int_0^1 \left\{ \left| x\frac{\mathrm{d}k''_n(x)}{\mathrm{d}x} \right|^2 + n(n+1)|k''_n(x)|^2 \right\} \mathrm{d}x + (n+1)|k''_n(1)|^2 \right) \;, \tag{0.97b}$$

$$\overline{\mathcal{E}}''_n = \frac{1}{2} \int_0^1 \Bigg\{ \bar{\lambda}(x) \left| x\frac{\mathrm{d}h''_n(x)}{\mathrm{d}x} + 2h''_n(x) - n(n+1)\ell''_n(x) \right|^2 + \bar{\mu}(x) \Bigg[2 \left| x\frac{\mathrm{d}h''_n(x)}{\mathrm{d}x} \right|^2 + 4|h''_n(x)|^2 - 2n(n+1)\left[h''_n(x)^* \ell''_n(x) + h''_n(x)\ell''_n(x)^* \right] + n(n+1)\Bigg[\left| x\frac{\mathrm{d}\ell''_n(x)}{\mathrm{d}x} - \ell''_n(x) + h''_n(x) \right|^2 + 2(n^2+n-1)|\ell''_n(x)|^2 \Bigg]\Bigg] \Bigg\} \mathrm{d}x \;, \tag{0.97c}$$

$$\overline{\mathcal{K}}''_n = \frac{1}{2} \frac{a|\omega|^2}{g(a)} \int_0^1 \left\{ |h''_n(x)|^2 + n(n+1)|\ell''_n(x)|^2 \right\} \bar{\rho}(x) x^2 \mathrm{d}x \tag{0.97d}$$

and for toroidal surface forces one has

$$\overline{\mathcal{E}}^{(\mathrm{t})}_n = \frac{1}{2} \int_0^1 \bar{\mu}(x) \left\{ \left| x\frac{\mathrm{d}f_n(x)}{\mathrm{d}x} - f_n(x) \right|^2 + (n+2)(n-1)|f_n(x)|^2 \right\} \mathrm{d}x \;, \tag{0.98a}$$

$$\overline{\mathcal{K}}^{(\mathrm{t})}_n = \frac{1}{2} \frac{a|\omega|^2}{g(a)} \int_0^1 |f_n(x)|^2 \bar{\rho}(x) x^2 \mathrm{d}x \;. \tag{0.98b}$$

0.2.4 Love-Shida numbers in case of a fluid surface

In the case of a fluid surface the shear stress vanishes there. We, therefore, have only body tide and load Love-Shida numbers. Coefficients for tangential forces, on the other hand, do not exist. The boundary conditions (0.8) in this case amount for

$$\tfrac{g}{4\pi G a} \mathrm{y}_{\mathrm{L},1}\big|_{r=a-0} - \tfrac{\rho\omega^2}{n} \mathrm{y}_{\mathrm{L},2}\big|_{r=a-0} - \tfrac{\rho}{n} \mathrm{y}_{\mathrm{L},3}\big|_{r=a-0} = \sqrt{a}\,\eta_{nm} \;, \tag{0.99a}$$

$$y_{\mathrm{L},4}\big|_{r=a-0} = 4\pi G a^{3/2}\,\tau_{nm} + (2n+1)\sqrt{a}\,C_{\mathrm{P}}\ . \tag{0.99b}$$

From this and the definitions (0.76) and (0.78) we derive the following form of the two linear independent solution vectors at the surface:

$$\mathbf{y} = \sqrt{a}\,C_{\mathrm{P}}\begin{pmatrix} \frac{4\pi G\rho a}{g}\tilde{h}_n \\ \frac{a}{g}n\tilde{\ell}_n \\ n(1+\tilde{k}_n) \\ 2n+1 \end{pmatrix}\ , \qquad \mathbf{y}' = \frac{4\pi G a^{3/2}}{2n+1}\tau_{nm}\begin{pmatrix} \frac{4\pi G\rho a}{g}\tilde{h}'_n \\ \frac{a}{g}n\tilde{\ell}'_n \\ n(1+\tilde{k}'_n) \\ 2n+1 \end{pmatrix}\ , \tag{100}$$

where the tilde denotes the fluid case of Love-Shida numbers. The boundary condition (0.99a) then yields the relations

$$\tilde{h}_n - \frac{a\omega^2}{g(a)}\tilde{\ell}_n - \tilde{k}_n \;=\; 1\ , \tag{0.101a}$$

$$\tilde{h}'_n - \frac{a\omega^2}{g(a)}\tilde{\ell}'_n - \tilde{k}'_n \;=\; 1 - (2n+1)\frac{g(a)}{4\pi G\rho(a)a}\ . \tag{0.101b}$$

Applying equation (56) to the two solutions (100) we again arrive at equation (0.88a) now for the fluid Love numbers

$$\tilde{k}'_n = \tilde{k}_n - \tilde{h}_n\ . \tag{102}$$

0.2.5 The special case $n = 1$

In the case of harmonic degree $n = 1$ there is a special exact solution corrsponding to rigid translations of the earth as a whole. This translationary exact solution is given by the spheroidal solution vector

$$\mathbf{y}_{\mathrm{S\,trans}} = \sqrt{r}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ g(r)-\omega^2 r \\ -3\omega^2 r \end{pmatrix}\ . \tag{103}$$

With aid of the Poisson equation $rg'(r) + 2g(r) = 4\pi G\rho r$ it may be checked that (103) indeed satisfies equation (20). However, this solution is physically meaningless for $\omega \neq 0$ because the inertia forces for this motion were supplied by an additional potential $\varphi = -\omega^2 r Y_{1m}(\vartheta,\chi)\exp(i\omega t)$ for which there is no physical reason in an inertial system. In a liquid layer this solution has continuation

$$\mathbf{y}_{\mathrm{L\,trans}} = \sqrt{r}\begin{pmatrix} 4\pi G\rho(r)r \\ r \\ g(r)-\omega^2 r \\ -3\omega^2 r \end{pmatrix}\ , \tag{104}$$

which satisfies (31). If this solution is multiplied by the constant $-\frac{g(a)}{a\omega^2}$ it has the form of the vector $\mathbf{y}$ in (87) at the surface of the earth and therefore satisfies the surface boundary conditions of body tides. From this we may conclude

$$h_1(r) \;=\; -\frac{g(a)}{a\omega^2}\ , \tag{0.105a}$$

$$\ell_1(r) \;=\; -\frac{g(a)}{a\omega^2}\ , \tag{0.105b}$$

$$k_1(r) \;=\; -\frac{g(r)}{a\omega^2}\ . \tag{0.105c}$$

From equations (0.88a) and (0.88b) we further conclude that

$$k'_1 \;=\; 0\ , \tag{0.106a}$$

$$k''_1 \;=\; -\frac{g}{a\omega^2} \tag{0.106b}$$

must hold at the surface of the earth as well.

Up to now we have done all calculations in an inertial system. However, for body tides, it is customary to give the Love-Shida numbers in the center of mass system. If we transform to a coordinate system the origin of which moves by a vector

$$\boldsymbol{s}(t) = s_\circ\exp(i\omega t)(\boldsymbol{e}_r + \nabla_1)Y_{1m}(\vartheta,\chi)\ , \tag{107}$$

the displacements and the incremental field transform to

$$H_1^{(\mathrm{tr})}(r) \;=\; H_1(r) - s_\circ \;, \tag{0.108a}$$

$$T_1^{(\mathrm{tr})}(r) \;=\; T_1(r) - s_\circ \;, \tag{0.108b}$$

$$R_1^{(\mathrm{tr})}(r) \;=\; R_1(r) - s_\circ(g(r) - r\omega^2) \;, \tag{0.108c}$$

while the toroidal displacement W_1 is not affected.

The solution $\mathbf{y}_{\mathrm{S}_{\mathrm{trans}}}$ satisfies the boundary conditions for body tides and, therefore, body tides of harmonic degree $n = 1$ have constant displacement coefficients $H_1(r) = H_1(a) = T_1(r) = T_1(a)$. Hence the center of mass in the case of body tides is given by the coefficient $s_\circ = H_1$. With the above transformation the body tide Love-Shida numbers in the center of mass system then become

$$h_1^{(\mathrm{CM})}(r) \;=\; 0 \;, \tag{0.109a}$$

$$\ell_1^{(\mathrm{CM})}(r) \;=\; 0 \;, \tag{0.109b}$$

$$k_1^{(\mathrm{CM})}(r) \;=\; -\frac{r}{a} \;. \tag{0.109c}$$

For loading tides, the center of mass of the earth together with the loading masses does not move because of conservation of linear momentum. The same is true for tangential forcing if the forces are exerted for instance by winds or ocean currents and the air and water masses are included in the definition of the center of mass. To do so is convenient since e.g. the motion of a satelite which might be used to measure the Love-Shida numbers does not seperate between the mass of the earth and the loading masses on the earth.

0.2.6 Asymptotic solutions for large harmonic degrees

Since meteorological as well as — due to topography — oceanic and even glacial loads have considerable components with large harmonic degrees in there expansion, for which numerical solutions become unstable, there is need for asymptotic solutions of the differential equations (15) and (20) for large n.

For large n the differential equation systems (15) and (20) show two general features. Firstly the solutions which are finite at the origin decay roughly as $(r/a)^n$ with increasing deepness under the surface ($r = a$) and secondly their eigenfrequencies behave asymptotically proportional to n. For this reason we make the transformation

$$r = a\,\exp\left(\frac{z}{n+1}\right) \tag{110}$$

and define

$$\lim_{n\to\infty} \frac{\omega}{n+1} = \sqrt{\frac{\xi\mu(a)}{a^2\rho(a)}} \;. \tag{111}$$

That is to say, we set $\xi = 0$ when tides generated by external forces with certain periods are studied. However, when eigenmodes of the system are studied, $\xi \neq 0$ is the quantity to be determined.

With aid of the definitions (110) and (111) the linear differential eqaution system (20) for spheroidal modes for large n asymptotically becomes

$$\frac{\mathrm{d}\mathbf{y}_\mathrm{S}}{\mathrm{d}z} = \mathbf{A}_{\mathrm{S},\infty}\mathbf{y}_\mathrm{S} + \mathcal{O}\left(\frac{1}{n+1}\right) \tag{112}$$

with a constant Matrix

$$\mathbf{A}_{\mathrm{S},\infty} = \begin{pmatrix} 0 & \frac{1}{\lambda+2\mu} & \frac{\lambda}{\lambda+2\mu} & 0 & 0 & 0 \\ -\xi\mu & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & \frac{1}{\mu} & 0 & 0 \\ 0 & \frac{-\lambda}{\lambda+2\mu} & \frac{4\mu(\lambda+\mu)}{\lambda+2\mu} - \xi\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ \gamma & 0 & -\gamma & 0 & 0 & 1 \end{pmatrix} , \tag{113}$$

where λ and μ are the surface values $\lambda = \lambda(a)$, $\mu = \mu(a)$ and we have introduced the abbreviation

$$\gamma \;\stackrel{\mathrm{def}}{=}\; 4\pi G a\rho(a) \;. \tag{114}$$

The matrix $\mathbf{A}_{\mathrm{S},\infty}$ has for $\xi \neq 0$ eigenvalues $\alpha_1 = 1$, $\alpha_2 = \sqrt{1 - \frac{\xi\mu}{\lambda+2\mu}}$, $\alpha_3 = \sqrt{1-\xi}$, $\alpha_4 = -\sqrt{1-\xi}$, $\alpha_5 = -\sqrt{1 - \frac{\xi\mu}{\lambda+2\mu}}$, $\alpha_6 = -1$ and eigenvectors

$$\mathbf{c}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} , \qquad \mathbf{c}_2 = \begin{pmatrix} \sqrt{1-\frac{\xi\mu}{\lambda+2\mu}} \\ (2-\xi)\mu \\ 1 \\ 2\mu\sqrt{1-\frac{\xi\mu}{\lambda+2\mu}} \\ 0 \\ \gamma \end{pmatrix} , \ \mathbf{c}_3 = \begin{pmatrix} 1 \\ 2\mu\sqrt{1-\xi} \\ \sqrt{1-\xi} \\ (2-\xi)\mu \\ 0 \\ -\gamma \end{pmatrix} ,$$
$$\mathbf{c}_4 = \begin{pmatrix} 1 \\ -2\mu\sqrt{1-\xi} \\ -\sqrt{1-\xi} \\ (2-\xi)\mu \\ 0 \\ -\gamma \end{pmatrix} , \ \mathbf{c}_5 = \begin{pmatrix} \sqrt{1-\frac{\xi\mu}{\lambda+2\mu}} \\ (\xi-2)\mu \\ -1 \\ 2\mu\sqrt{1-\frac{\xi\mu}{\lambda+2\mu}} \\ 0 \\ -\gamma \end{pmatrix} , \qquad \mathbf{c}_6 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} . \tag{115}$$

The general solution which is finite at the origin with them reads

$$\mathbf{y}_\mathrm{S}(z) = \sum_{i=1}^{3} C_i \, \exp(\alpha_i z) \, \mathbf{c}_i + \mathcal{O}\left(\frac{1}{n+1}\right) \tag{116}$$

and from the boundary conditions (0.73a), (0.73b) and (0.73d) with $\eta_{nm} = 0$, $\theta_{nm} = 0$, $\tau_{nm} = 0$ we conclude that the eigenfrequency of the spheroidal mode is asymptotically given by the value of ξ satisfying

$$\det \begin{pmatrix} 0 & (2-\xi)\mu & 2\mu\sqrt{1-\xi} \\ 0 & 2\mu\sqrt{1-\frac{\xi\mu}{\lambda+2\mu}} & (2-\xi)\mu \\ 1 & \gamma & -\gamma \end{pmatrix} = 0 \, . \tag{117}$$

This is the zero of the cubic equation

$$\xi^3 - 8\xi^2 + 8(3-2\nu)\xi - 16(1-\nu) = 0 \tag{118}$$

with $\nu = \frac{\mu}{\lambda+2\mu}$ and $0 < \xi < 1$. For a proportion ν ranging from zero to $1/2$ the value ξ for the eigenfrequency varies between 0.912622 and 0.763932.

For $\xi = 0$ the eigenvalues of the matrix $\mathbf{A}_{\mathrm{S},\infty}$ are ± 1 but $\mathbf{A}_{\mathrm{S},\infty}$ is no longer diagonalizable. Corresponding to the eigenvalue $+1$ there are two eigenvectors $\mathbf{c}_1^\circ$, $\mathbf{c}_2^\circ$ and a characteristic vector $\mathbf{c}_3^\circ$ satisfying $\mathbf{A}_{\mathrm{S},\infty}\mathbf{c}_3^\circ = \mathbf{c}_2^\circ + \mathbf{c}_3^\circ$ which explicitly read

$$\mathbf{c}_1^\circ = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} , \quad \mathbf{c}_2^\circ = \begin{pmatrix} 1 \\ 2\mu \\ 1 \\ 2\mu \\ 0 \\ -\gamma\frac{\lambda+3\mu}{\lambda+\mu} \end{pmatrix} , \quad \mathbf{c}_3^\circ = \begin{pmatrix} \frac{-\mu}{\lambda+\mu} \\ 0 \\ \frac{\lambda+2\mu}{\lambda+\mu} \\ 2\mu \\ 0 \\ 0 \end{pmatrix} . \tag{119}$$

With them the general solution finite at the origin is

$$\mathbf{y}_\mathrm{S}(z) = \exp(z) \left\{ C_1\mathbf{c}_1^\circ + [C_2 + (n+1)C_3 z]\mathbf{c}_2^\circ + C_3\mathbf{c}_3^\circ \right\} + \mathcal{O}\left(\frac{1}{n+1}\right) \, . \tag{120}$$

From equation (87) one concludes that for body tides one must have $C_1 = (2n+1)g$, $C_2 = C_3 = 0$ and, therefore, the asymptotical values of the body tide Love-Shida numbers for large n are given by

$$h_n = \mathcal{O}(1) \, , \qquad \ell_n = \mathcal{O}\left(\frac{1}{n}\right) \, , \qquad k_n = \mathcal{O}(1) \, . \tag{0.121a}$$

Likewise, for loading tides one has $C_1 = (2n+1)g\{1 - \frac{g\rho a(\lambda+3\mu)}{2(n+1)\mu(\lambda+\mu)}\}$, $C_2 = -C_3 = \frac{-(2n+1)g^2}{8\pi G(n+1)\mu}$ and

$$h_n' = \mathcal{O}(1) \, , \qquad \ell_n' = \mathcal{O}\left(\frac{1}{n}\right) \, , \qquad k_n' = \mathcal{O}(1) \, , \tag{0.121b}$$

and for tangential surface forces one has $C_1 = C_2 = 0$, $C_3 = \frac{(2n+1)g^2}{8\pi G n(n+1)\mu}$

$$h_n'' = \frac{-g^2}{4\pi G(\lambda+\mu)n} + \mathcal{O}\left(\frac{1}{n^2}\right) , \quad \ell_n'' = \frac{-g^2(\lambda+2\mu)}{4\pi G\mu(\lambda+\mu)n^2} + \mathcal{O}\left(\frac{1}{n^3}\right) , \quad k_n'' = \mathcal{O}\left(\frac{1}{n^2}\right) , \tag{0.121c}$$

For toroidal modes one could use the explicit solution (58) to gain the asymptotes directly, but it is easier to handle it in the same way as the case of spheroidal modes using the matrix

$$\mathbf{A}_{\mathrm{T},\infty} = \lim_{n\to\infty} \frac{1}{n+1}\mathbf{A}_{\mathrm{T}} = \begin{pmatrix} 0 & \frac{1}{\mu} \\ \mu(1-\xi) & 0 \end{pmatrix} , \tag{122}$$

where $\mu = \mu(a)$. The equation

$$\frac{\mathrm{d}\mathbf{y}_{\mathrm{T}}}{\mathrm{d}z} = \mathbf{A}_{\mathrm{T},\infty}\mathbf{y}_{\mathrm{T}} + \mathcal{O}\left(\frac{1}{n+1}\right) \tag{123}$$

then yields the asymptotic solution

$$\mathbf{y}_{\mathrm{T}}(r) = C\,\exp(\sqrt{1-\xi}\,z)\begin{pmatrix} 1 \\ \mu\sqrt{1-\xi} \end{pmatrix} + \mathcal{O}\left(\frac{1}{n+1}\right) . \tag{124}$$

The boundary condition (0.73c) with $\zeta_{nm} = 0$ then amounts to $\sqrt{1-\xi} = 0$ and hence the eigenfrequency for toroidal modes is up to leading order asymptotically given by the value $\xi = 1$.

The solution (124) with $\xi = 0$ together with the definition (86) then yuields the asymptotical value

$$f_n = \frac{g^2}{4\pi G\mu n} + \mathcal{O}\left(\frac{1}{n^2}\right) \tag{125}$$

for the toroidal forcing coefficient.

The case of a fluid surface is treated likewise. Equation (31) becomes

$$\frac{\mathrm{d}\mathbf{y}_{\mathrm{L}}}{\mathrm{d}z} = \mathbf{A}_{\mathrm{L},\infty}\mathbf{y}_{\mathrm{L}} + \mathcal{O}\left(\frac{1}{n+1}\right) \tag{126}$$

with

$$\mathbf{A}_{\mathrm{L},\infty} = \begin{pmatrix} 0 & 4\pi G\rho & 0 & 0 \\ \frac{1}{4\pi G\rho} & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 \\ 1 & -4\pi G\rho & 0 & 1 \end{pmatrix} , \tag{127}$$

where now we have defined

$$\lim_{n\to\infty} \frac{\omega^2}{n} = \frac{g(a)}{a}\tilde{\xi} . \tag{128}$$

The matrix $\mathbf{A}_{\mathrm{L},\infty}$ has eigenvalues $\tilde{\alpha}_1 = \tilde{\alpha}_2 = 1$, $\tilde{\alpha}_3 = \tilde{\alpha}_4 = -1$ and corresponding eigenvectors

$$\tilde{\mathbf{c}}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 2 \end{pmatrix} , \quad \tilde{\mathbf{c}}_2 = \begin{pmatrix} 1 \\ \frac{1}{4\pi G\rho} \\ 1 \\ 1 \end{pmatrix} , \quad \tilde{\mathbf{c}}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} , \quad \tilde{\mathbf{c}}_4 = \begin{pmatrix} 1 \\ \frac{-1}{4\pi G\rho} \\ -1 \\ -1 \end{pmatrix} . \tag{129}$$

The general solution finite at the origin in this case is

$$\mathbf{y}_{\mathrm{L}}(z) = \sum_{i=1}^{2} \tilde{C}_i\,\exp(\tilde{\alpha}_i z)\,\tilde{\mathbf{c}}_i + \mathcal{O}\left(\frac{1}{n+1}\right) \tag{130}$$

and equation (0.99a) in conjunction with (128) yields the value $\tilde{\xi} = 1$ for the asymptotic behaviour of the eigenfrequency. Comparison with (100) and (0.99a) shows that we must have $\tilde{C}_1 = \sqrt{a}\,C_{\mathrm{P}}(n + 1/2 - 2\pi G\rho a/g)$, $\tilde{C}_2 = 4\pi G\rho a^{3/2}C_{\mathrm{P}}/g$ for body and $\tilde{C}_1 = 4\pi G a^{3/2}\tau_{nm}$, $\tilde{C}_2 = -4\pi G a^{3/2}\tau_{nm}$ for loading tides respectively and therefore yields the asymptotic values

$$\tilde{h}_n = \mathcal{O}(1) , \qquad \tilde{\ell}_n = \mathcal{O}\left(\frac{1}{n}\right) , \qquad \tilde{k}_n = \mathcal{O}\left(\frac{1}{n}\right) , \tag{0.131a}$$

$$\tilde{h}_n' = -\frac{(2n+1)g}{4\pi G\rho a} + \mathcal{O}(1) , \ \tilde{\ell}_n' = -\frac{g}{2\pi G\rho a} + \mathcal{O}\left(\frac{1}{n}\right) , \ \tilde{k}_n' = -1 + \mathcal{O}\left(\frac{1}{n}\right) . \tag{0.131b}$$

Bibliography